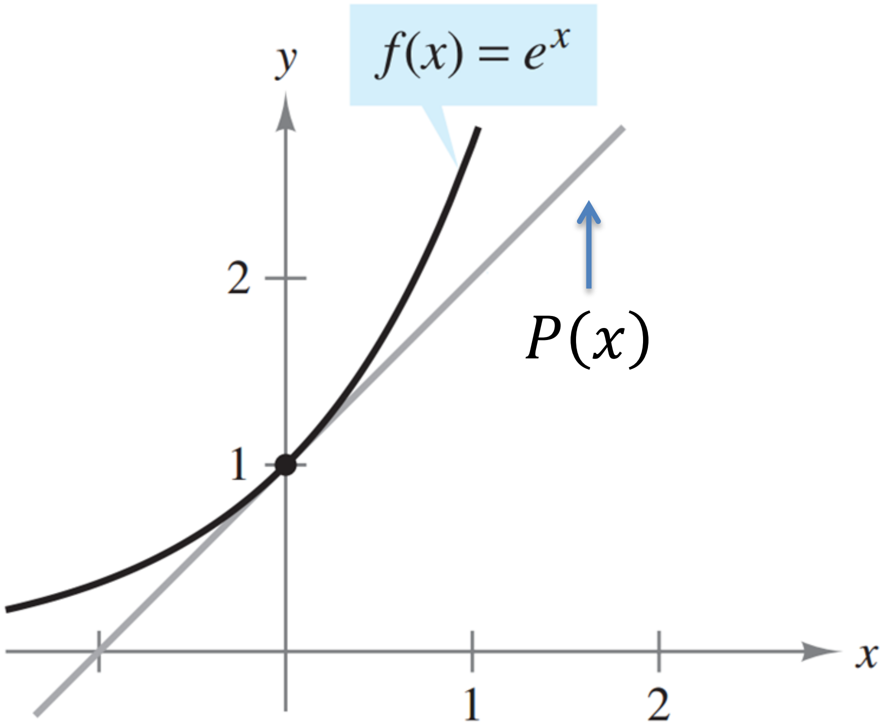
Taylor and Maclaurin Series

Approximating functions using Polynomials.

Approximating $f(x) = e^x$ near $x = 0$

In order to approximate the function $f(x) = e^x$ near $x = 0$, we can use the tangent line **(The Linear Approximation).**



$$f(0) = 1$$

$$f'(x) = e^x$$

$$f'(0) = e^0 = 1$$

$$P(x) - 1 = 1(x - 0)$$

$$P(x) = 1 + x$$

$$P_1(x) = 1 + x$$

The 1st Degree Polynomial Approximation

$$P_1(0) = f(0) = 1 \qquad P_1{}'(0) = f'(0) = 1$$

A Better Approximation to $f(x) = e^x$

As we move away from $x = 0$, our line is not a good approximation to e^x. How could we get a better approximation?

To get a better approximation, we can create a function, $P_2(x)$, that also has the same concavity or 2nd Derivative as $f(x) = e^x$.

$f''(x) = e^x$

$P_2(0) = f(0) = 1$ $\quad$ $P_2'(0) = f'(0) = 1$ $\quad$ $P_2''(0) = f''(0) = 1$

$$P_2(x) = 1 + x + \frac{x^2}{2}$$

The 2nd Degree Polynomial Approximation

Graph

The next step, in order to get an even better approximation is to allow their third derivatives to equal one another. This can be done with the following equation:

$$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

Graph

In general, the function e^x can be found by the following power series:

$$P_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

$$\left(e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \right)$$

For what values of x will $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$?

For all x values in the **interval of convergence**!!

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \qquad \lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| \qquad \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right|$$

$$\lim_{n\to\infty} \left|\frac{x}{n+1}\right| < 1$$

This holds for all values of x. Therefore, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ everywhere!

Graph

Maclaurin Polynomial Approximation

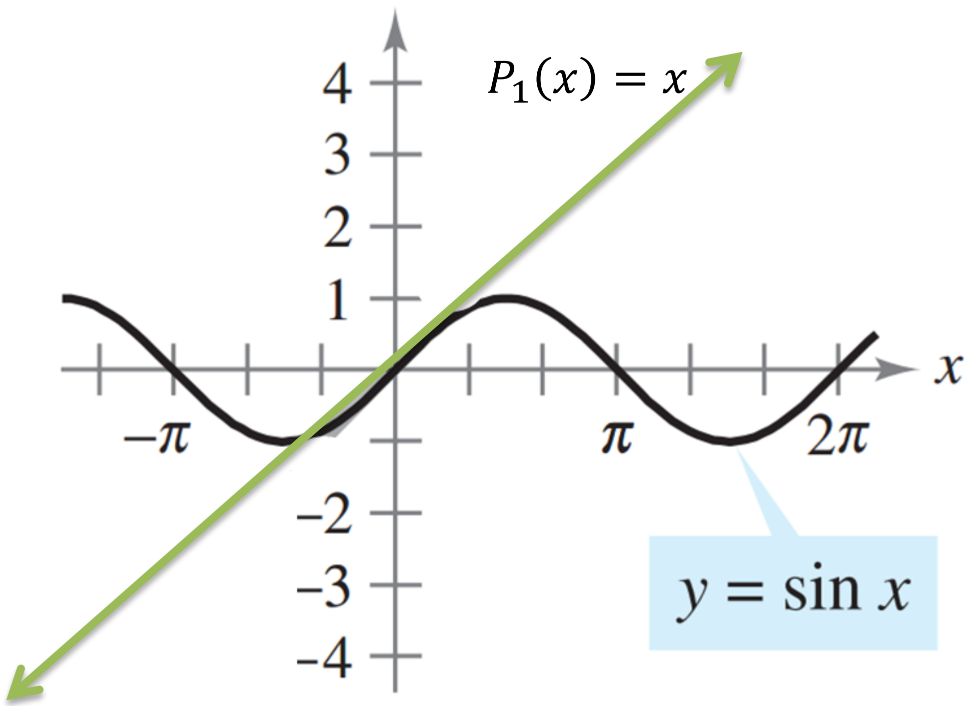
When developing a polynomial to estimate a function centered at $c = 0$, you can use the following:

$$P_n(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^n(0)}{n!}x^n$$

$$P_n(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n$$

***A Maclaurin Polynomial is a Taylor Polynomial centered at $c = 0$.**

Develop a Maclaurin Polynomial Approximation for the graph of $f(x) = \sin x$.



$$P_n(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$

$$P_0(x) = \frac{\sin(0)}{0!} x^0 = \frac{0}{1} \cdot 1 = 0$$

1st Degree Polynomial Approximation

$$P_1(x) = 0 + \frac{\cos(0)}{1!} x^1 \ \ = x$$

$$P_1(x) = x$$

(Tangent line at $x = 0$)

$P_1(0) = f(0)$ $\qquad P_1{}'(0) = f'(0)$ $\qquad P_1(x) = x$

2nd Degree Polynomial Approximation

$$P_n(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$

$$f''(x) = -\sin x$$

$$f''(0) = -\sin 0 = 0$$

$$P_2(x) = 0 + x +$$

$$P_2(x) = x$$

3rd Degree Polynomial Approximation

$$f'''(x) = -\cos x$$

$$f'''(o) = -\cos 0 = -1$$

$$P_3(x) = x + ?$$

$$P_3(x) = x - \frac{x^3}{3!}$$

4th Degree Polynomial Approximation

$$P_n(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$

$$P_4(x) = x - \frac{x^3}{3!} + \cdots$$

$$f^4(x) = \sin x$$

$$f''(0) = \sin 0 \quad = 0$$

$$P_4(x) = x - \frac{x^3}{3!}$$

5th Degree Polynomial Approximation

$$f^5(x) = \cos x$$

$$f^5(o) = \cos 0 = 1$$

$$P_5(x) = x \; - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Maclaurin Polynomial for $f(x) = \sin x$

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$P_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

For what values of x does $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$?

For all x values in the **interval of convergence**

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

Homework

Using the formula for a Maclaurin Polynomial, $P_n(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$
Write the first 4 nonzero terms of the polynomial approximation for $f(x) = \cos x$, $f(x) = \sin x$ and $f(x) = e^x$.

Try to find the pattern so you can come up with the formula for the n^{th} term of the series in each case.

Section 9.2 (2, 3, 22, 24, 31)

****This is very important for the BC Calculus test.****

$\cos x$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!}$$

$\sin x$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

e^x

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots + \frac{x^n}{n!}$$

Interval of Convergence for Power Series Representation of $\cos x$

$$P_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Problem 2 from homework

$$e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots$$

Can you come up with a formula for the nth term?

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Problem 24b, c

(b) Multiply each term of $f(x)$ by x.

$$g(x) = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{n+1}}{(n+1)!} + \cdots$$

(c) $g(x) = e^x - 1$

Developing a Taylor Series for a Composite Function

How do you develop a Taylor Series for a composite function like $\sin x^2$?

Find the Maclaurin series for $f(x) = \sin x^2$

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n$$

~~$f'(x) = 2x\cos x^2$~~

~~$f''(x) = -4x^2\sin x^2 + 2\cos x^2$~~

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(x^2) = (x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \frac{(x^2)^9}{9!} - \cdots + \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - \cdots + \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

GRAPH

Write the first three nonzero terms and then general term of the power series centered at 0 that will represent the following: $g(x) = \frac{e^x - 1}{x}$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots + \frac{x^n}{n!}$$

$$g(x) = \frac{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots + \frac{x^n}{n!} - 1}{x}$$

$$g(x) = 1 + \frac{x}{2} + \frac{x^2}{6} + \cdots + \frac{x^{n-1}}{n!}$$

Write the first three nonzero terms and then general term of the power series centered at 0 that will represent the following: $g(x) = 4x \cdot \sin 3x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(3x) = (3x) - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} + \frac{(3x)^9}{9!} - \cdots + \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!}$$

$$g(x) = (4x)(3x) - (4x)\frac{(3x)^3}{3!} + (4x)\frac{(3x)^5}{5!} - \cdots + (4x)\frac{(-1)^n (3x)^{2n+1}}{(2n+1)!}$$

Taylor Series Polynomials

How can we develop a Polynomial whose center is not $c = 0$?

Taylor and Maclaurin Series

Maclaurin Series

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n$$

Taylor Series

$$\sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n$$

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^n(c)}{n!}(x-c)^n$$

Finding the third degree Taylor Polynomial for $f(x) = \ln x$ centered at $c = 1$.

$$\sum_{n=0}^{\infty} \frac{f^{n}(c)}{n!}(x-c)^{n}$$

$f(x) = \ln x$ $\quad f(1) = 0$

$f'(x) = 1/x$ $\quad f'(1) = 1$

$f''(x) = -1/x^2$ $\quad f''(1) = -1$

$f'''(x) = 2/x^3$ $\quad f'''(1) = 2$

$$P_3(x) = \frac{0}{0!}(x-1)^0 + \frac{1}{1!}(x-1)^1 + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$$

$$P_3(x) = (x-1) - 1/2 \cdot (x-1)^2 + 1/3 \cdot (x-1)^3$$

GRAPH

Find the third order Taylor Polynomial for $f(x) = 2x^3 - 3x^2 + 4x - 5$ centered at $c = 1$.

$$\sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n$$

$$f(x) = 2x^3 - 3x^2 + 4x - 5 \qquad f(1) = -2$$

$$f'(x) = 6x^2 - 6x + 4 \qquad f'(1) = 4$$

$$f''(x) = 12x - 6 \qquad f''(1) = 6$$

$$f'''(x) = 12 \qquad f'''(1) = 12$$

$$P_3(x) = \frac{-2}{0!}(x-1)^0 + \frac{4}{1!}(x-1)^1 + \frac{6}{2!}(x-1)^2 + \frac{12}{3!}(x-1)^3$$

$$P_3(x) = -2 + 4 \cdot (x-1) + 3 \cdot (x-1)^2 + 2 \cdot (x-1)^3$$

$$P_3(x) = 2 \cdot (x-1)^3 + 3 \cdot (x-1)^2 + 4 \cdot (x-1) - 2$$

Now use the expand key on your calculator to verify that $P_3(x) = f(x)$ (Home > F2 > expand).

Find the Taylor Series generated by $f(x) = e^x$ at $c = 2$.

$$\sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n$$

$$f(x) = f'(x) = f''(x) = f'''(x) = f^4(x) = e^x$$

$$\frac{f(2)}{0!}(x-2)^0 + \frac{f'(2)}{1!}(x-2)^1 + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \ldots$$

$$e^2 + e^2 \cdot (x-2) + \frac{e^2}{2} \cdot (x-2)^2 + \frac{e^2}{6} \cdot (x-2)^3$$

$$+ \cdots + \frac{e^2}{n!}(x-2)^n$$

$$\sum_{k=0}^{\infty} \frac{e^2}{k!}(x-2)^k$$

Graph

Find the third Taylor polynomial for $f(x) = \sin x$, expanded about $c = \pi/6$.

$$\sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n$$

$$f(x) = \sin x \qquad f(\pi/6) = 1/2$$

$$f'(x) = \cos x \qquad f'(\pi/6) = \sqrt{3}/2$$

$$f''(x) = -\sin x \qquad f''(\pi/6) = -1/2$$

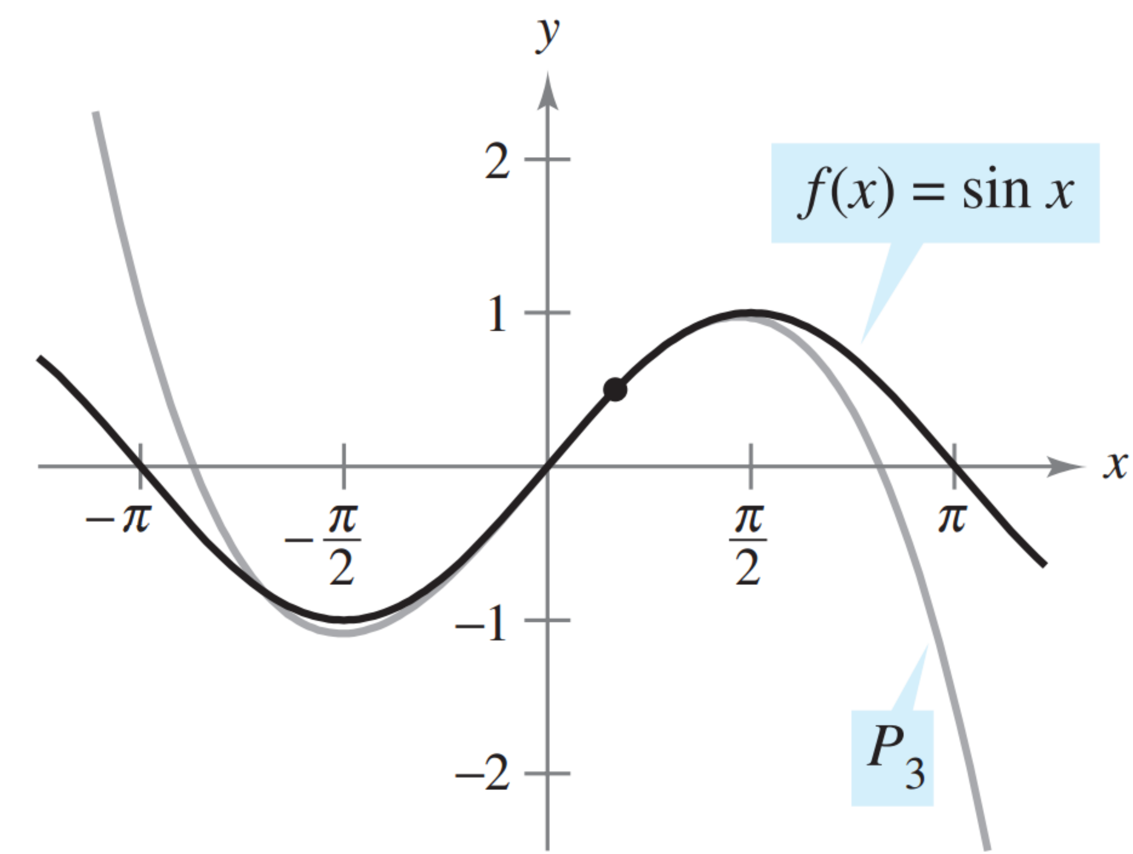
$$f'''(x) = -\cos x \qquad f'''(\pi/6) = -\sqrt{3}/2$$

$$P_3(x) = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) + \frac{f''\left(\frac{\pi}{6}\right)}{2!}\left(x - \frac{\pi}{6}\right)^2 + \frac{f'''\left(\frac{\pi}{6}\right)}{3!}\left(x - \frac{\pi}{6}\right)^3$$

$$P_3(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{2(2!)}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{2(3!)}\left(x - \frac{\pi}{6}\right)^3$$

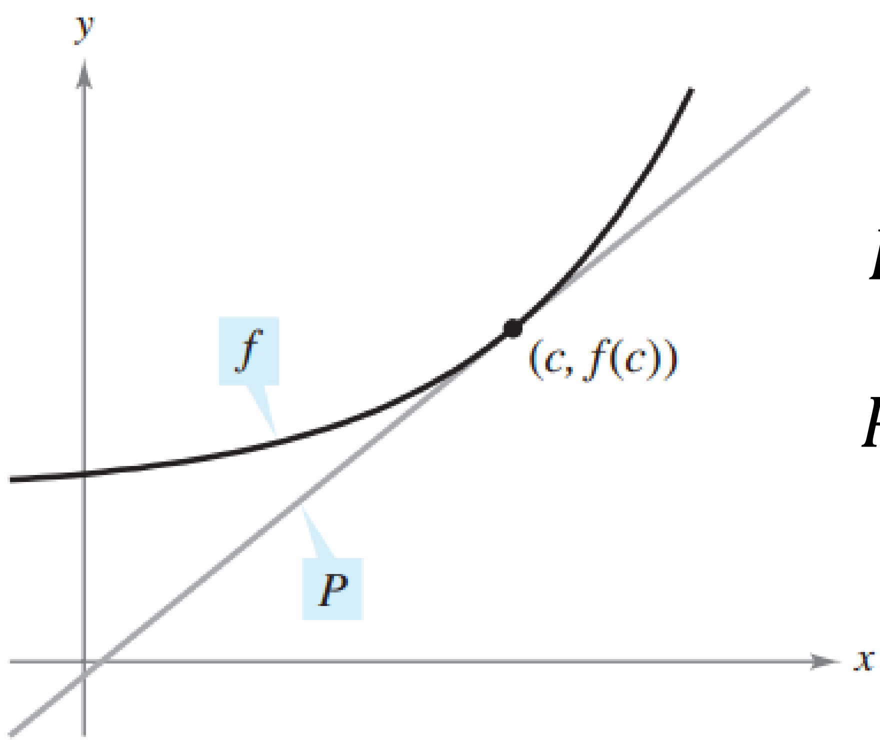
$$f(x) = \sin x$$

$$P_3(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{2(2!)}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{2(3!)}\left(x - \frac{\pi}{6}\right)^3$$



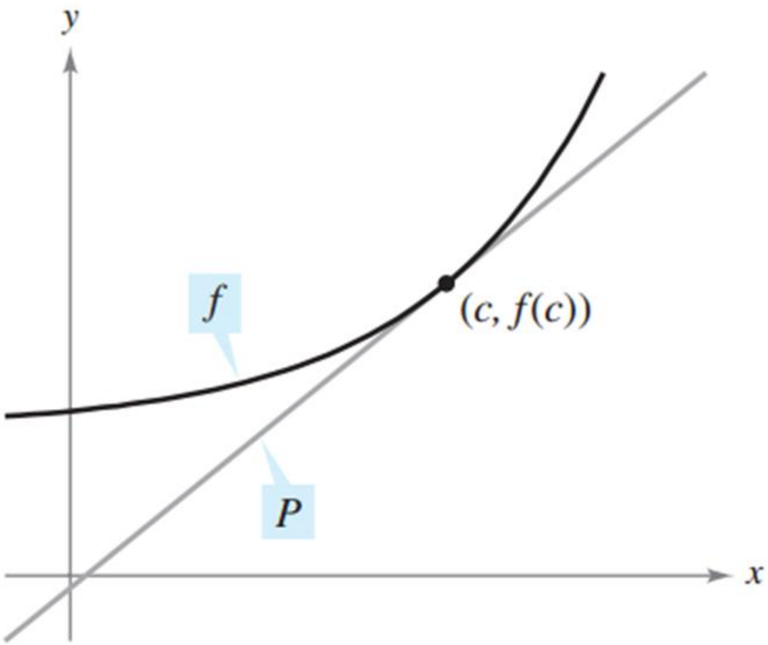
The General Formula for a Polynomial Approximation

What is the equation of the tangent line, $P(x)$, to a function $f(x)$ at the point c?



$$P(x) - f(c) = f'(c)(x - c)$$

$$P(x) = f(c) + f'(c)(x - c)$$



$$P_1(x) = f(c) + f'(c)(x - c)$$

$$x = c$$

$$P(c) = f(c)$$

$$P'(x) = f'(c) \quad (f'(c)\ constant)$$

$$P'(c) = f'(c)$$

In order to make $P(x)$ a better approximation for $f(x)$, I am going to add a term so that the second derivatives also equal one another.

$$P_2(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2}(x - c)^2$$

The 2nd Degree Polynomial Approximation

$$P_2(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2$$

$$x = c$$

$$P(c) = f(c)$$

First Derivative of $P(x)$

$$P'(x) = f'(c) + f''(c)(x-c)$$

$(f'(c)\ constant)$
$(f''(c)\ constant)$

$$P'(c) = f'(c)$$

Second Derivative of $P(x)$

$$P''(x) = f''(c)$$

$$P''(c) = f''(c)$$

Allowing the 3rd Derivatives to be Equal

$$P_3(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2}(x - c)^2 + \frac{f'''(c)}{6}(x - c)^3$$

The 3rd Degree Polynomial Approximation

$$x = c$$

$$P(c) = f(c)$$

$$P'(c) = f'(c) \qquad (f'(c)\ constant)$$

$$P''(c) = f''(c) \qquad (f''(c)\ constant)$$

$$P^3\ (c) = f^3(c)$$

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \frac{f'''(c)}{3!}(x - c)^3 + \cdots + \frac{f^n(c)}{n!}(x - c)^n$$

The Nth Degree Polynomial Approximation

Steps to Developing a Taylor Polynomial:

1. Write out the formula for a Taylor Polynomial
$$\sum_{n=0}^{\infty} \frac{f^n(c)}{n!} \cdot (x-c)^n$$
2. Write out the function and its derivatives.
3. Evaluate the derivatives at $x = c$.
4. Write out terms and plug in for
$$f^n(c),\ c \text{ and } n.$$
5. If possible, come up with a formula for the n^{th} term.
6. Graph the original function and the polynomial to be sure they are equal.

Homework (Finney/Demana)

Section 9.2: (13, 16, 20, 21, 23) (5, 8, 33)
Section 9.3: (6, 8, 9)

Try to do 13 two ways:
A: Developing a Taylor Polynomial
B: Using a Geometric Series

Taylor and Maclaurin Series

We can take this formula and use it to come up with a Power Series that will exactly represent many elementary functions including:

$$\sin x , \cos x , e^x$$

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^n(c)}{n!}(x-c)^n$$

Taylor Series

$$\sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n$$

Maclaurin Series

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n$$

Error in an Approximation