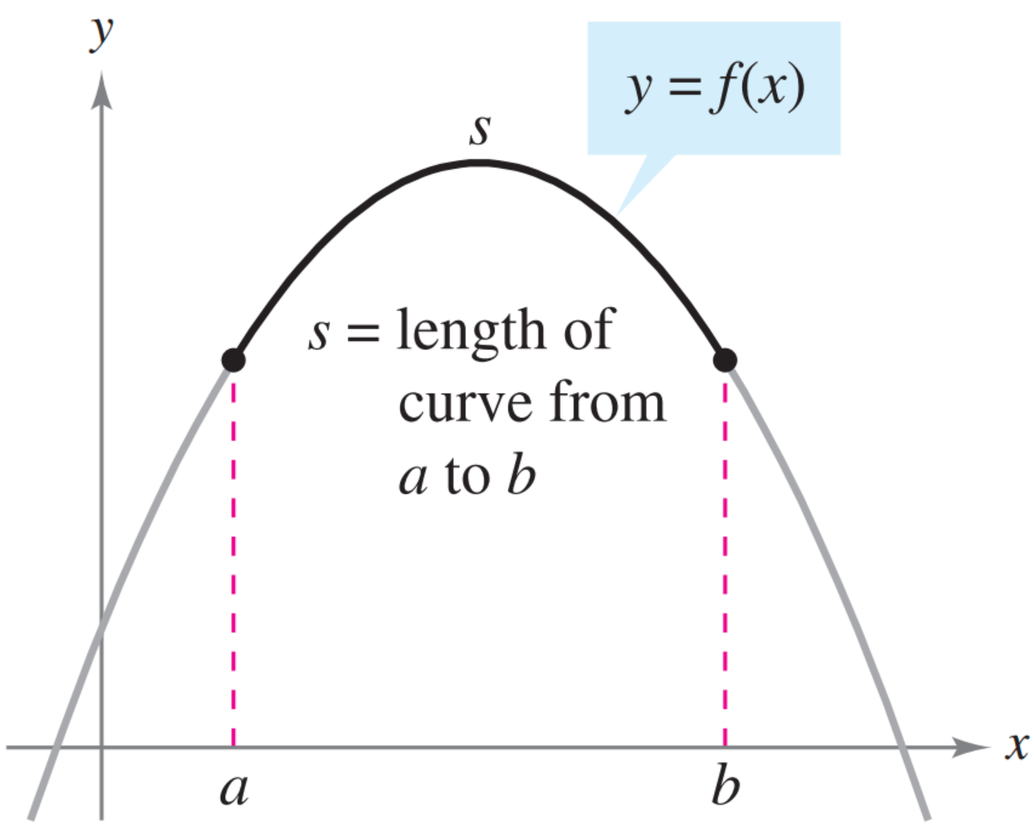
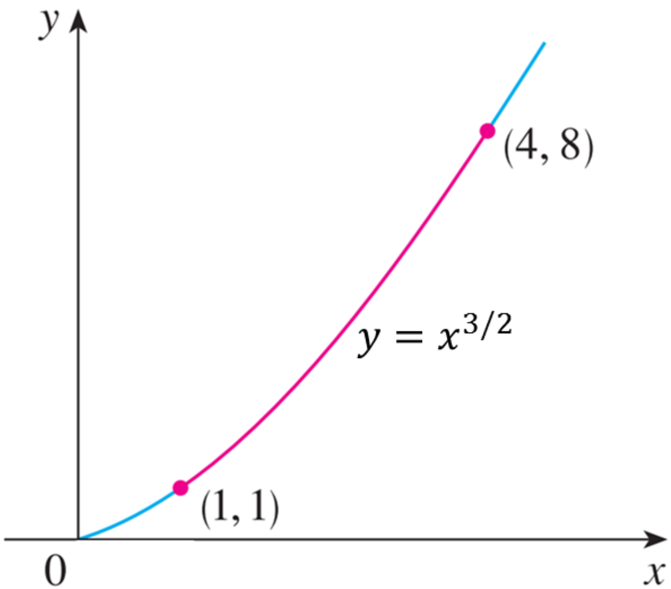
Arc Length

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$



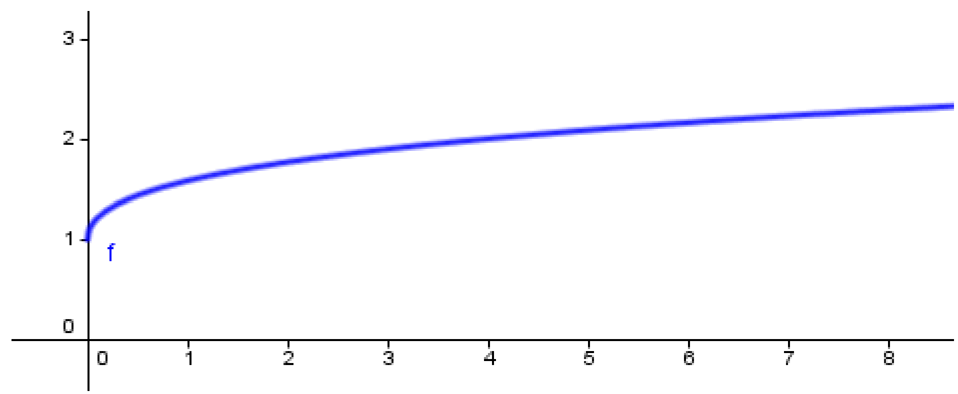
Find the length of the curve of $y = x^{3/2}$ from the points $(1, 1)$ to $(4, 8)$.



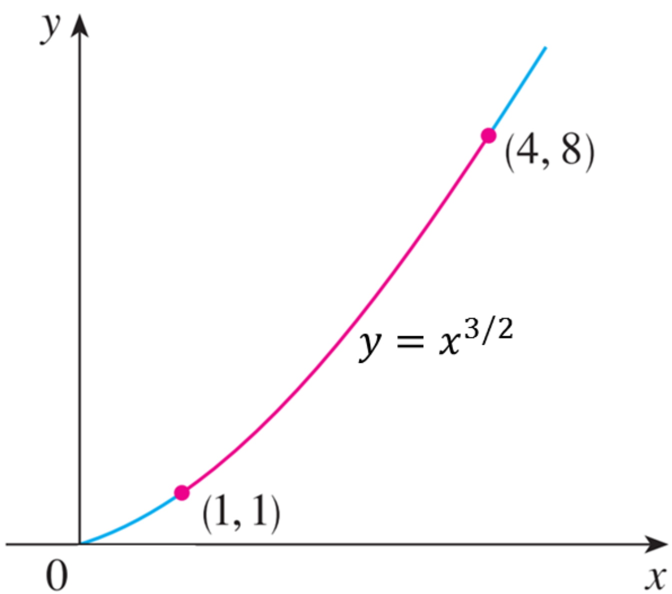
$$Arc\ Length = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$Arc\ Length = \int_1^4 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\ dx$$

$$g(x) = \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}$$



Find the length of the curve of $y = x^{3/2}$ from the points $(1, 1)$ to $(4, 8)$.



$$Arc\ Length = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$Arc\ Length = \int_1^4 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\ dx$$

$$= \int_1^4 \sqrt{1 + \frac{9}{4}x}\ dx$$

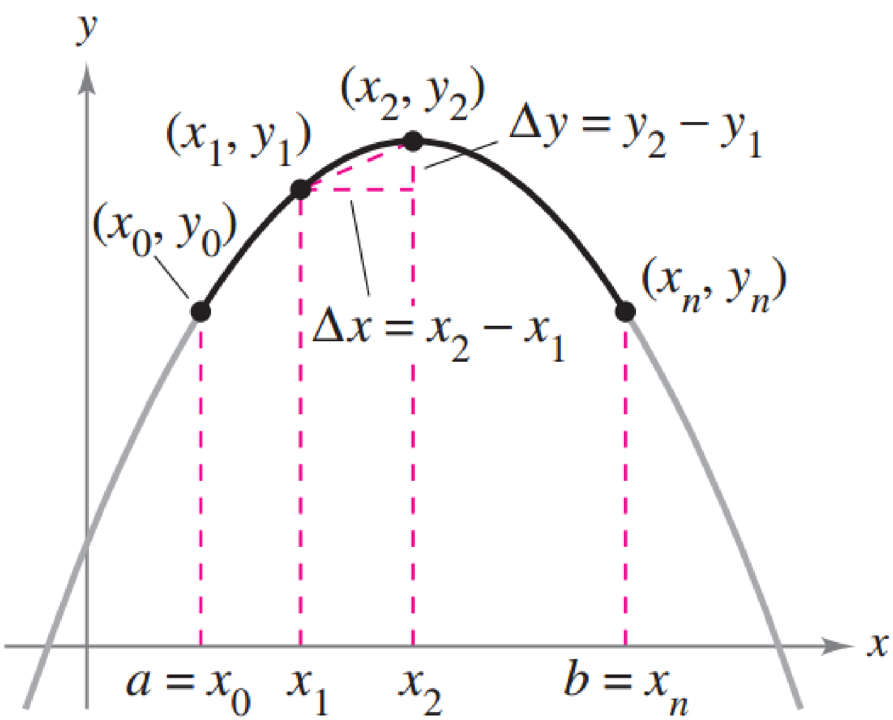
$$u = \frac{9}{4}x$$

$$du = \frac{9}{4}dx$$

$$= \frac{4}{9}\int_1^4 \frac{9}{4}\sqrt{1 + \frac{9}{4}x}\ dx \quad = \frac{4}{9}\int_{9/4}^{9} \sqrt{1 + u}\ du \quad = \frac{4}{9}\left[\frac{2}{3}(1 + u)^{3/2}\right]\Bigg|_{9/4}^{9} \approx 7.633705$$

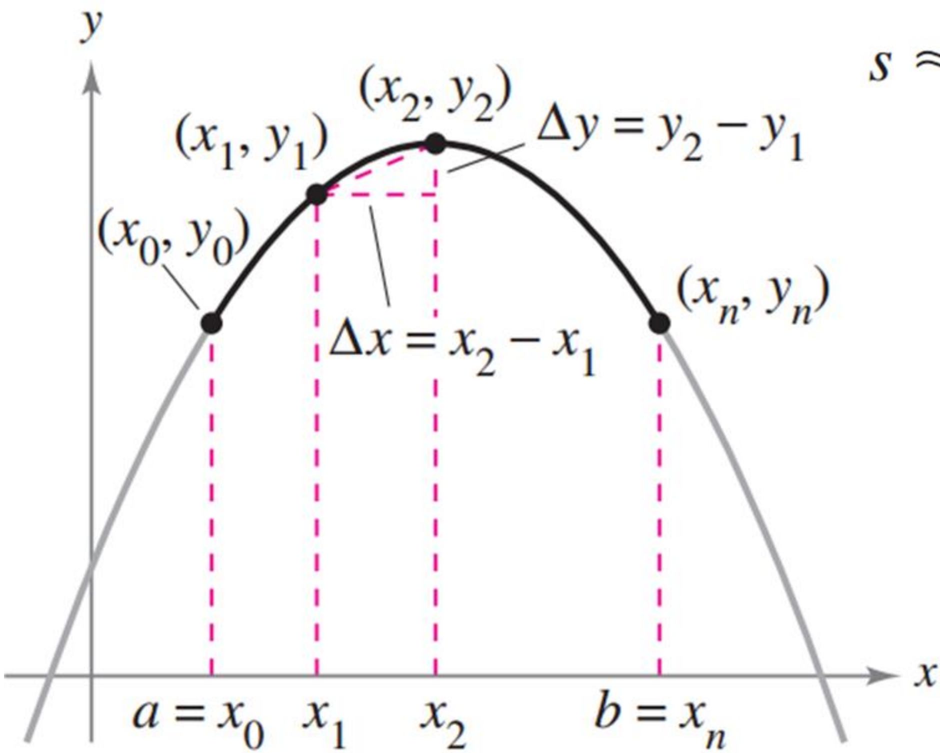
Where does the formula come from?

In order to find the length of a curve, then the curve must be rectifiable. A curve $(f(x))$ is rectifiable if its derivative $f'(x)$ exists on the interval in question.



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$s \approx \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$



$$s \approx \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

$$= \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$= \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2 (\Delta x_i)^2}$$

$$= \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} (\Delta x_i)$$

$$\lim_{n \to \infty} = \lim_{\|\Delta\| \to 0}$$

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} (\Delta x_i)$$

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} (\Delta x_i)$$

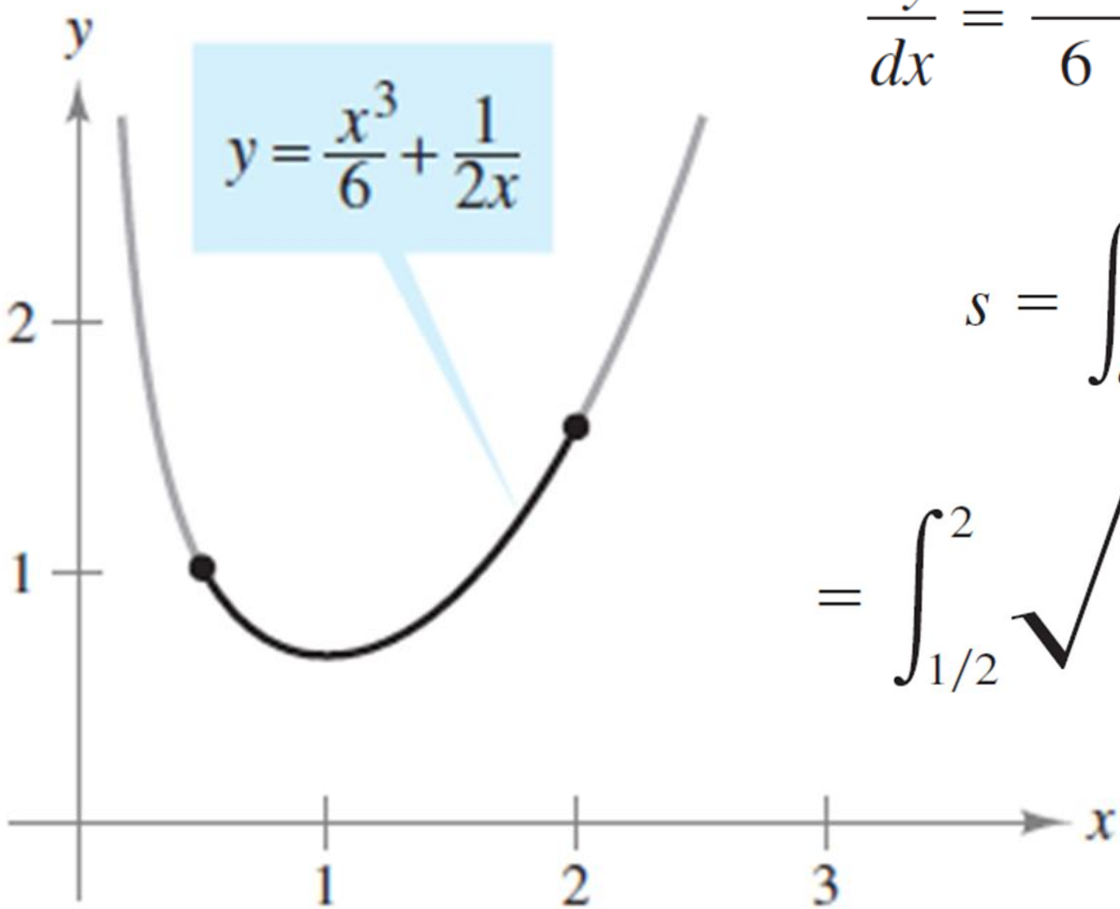
Since $f'(x)$ exists (the curve is rectifiable), according to the mean value theorem, there must be a point in the interval, c_i, such that $f'(c_i) =$ The Average Rate of Change on that Interval $\left(\frac{\Delta y_i}{\Delta x_i}\right)$.

$$= \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2} (\Delta x_i)$$

Now note that we have created an Infinite Reimann Sum for the function $g(x) = \sqrt{1 + [f'(x)]^2}$ from $a < x < b$. We know that this is equivalent to the area under the cruve from a to b, therefore:

$$s = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

What is the arc length from $\left[\frac{1}{2}, 2\right]$ of the line $y = \frac{x^3}{6} + \frac{1}{2x}$?

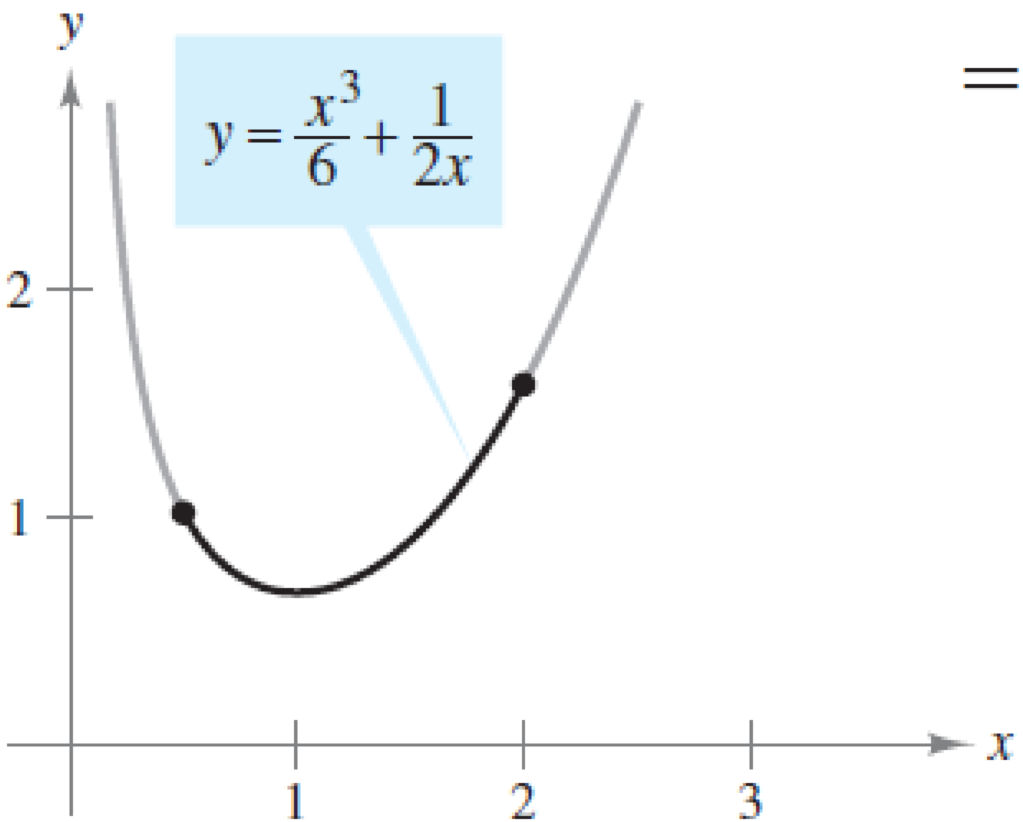


$$\frac{dy}{dx} = \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$$

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{1/2}^{2} \sqrt{1 + \left[\frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)\right]^2}\, dx$$

$$= \int_{1/2}^{2} \sqrt{\frac{1}{4}\left(x^4 + 2 + \frac{1}{x^4}\right)}\, dx$$



$$= \int_{1/2}^{2} \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right) dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{1}{x}\right]_{1/2}^{2}$$

$$= \frac{1}{2}\left(\frac{13}{6} + \frac{47}{24}\right)$$

$$= \frac{33}{16}$$

Memorizing the Formula dx

First just think about the distance formula as:

$$\sqrt{\boldsymbol{dx^2 + dy^2}}$$

Then multiply dy^2 by $\frac{dx^2}{dx^2}$ and factor our the dx^2.

$$\sqrt{\mathbf{1} + \left(\frac{\boldsymbol{dy}}{\boldsymbol{dx}}\right)^{\mathbf{2}}}\,\boldsymbol{dx}$$

Then remember to integrate from a to b.

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Memorizing the Formula dy

First just think about the distance formula as:

$$\sqrt{\boldsymbol{dx^2 + dy^2}}$$

Then multiply dx^2 by $\frac{dy^2}{dy^2}$ and factor our the dy^2.

$$\sqrt{\left(\frac{\boldsymbol{dx}}{\boldsymbol{dy}}\right)^{\mathbf{2}} + \mathbf{1}\ \boldsymbol{dx}}$$

Then integrate from c to d $(y - values)$.

$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dx$$

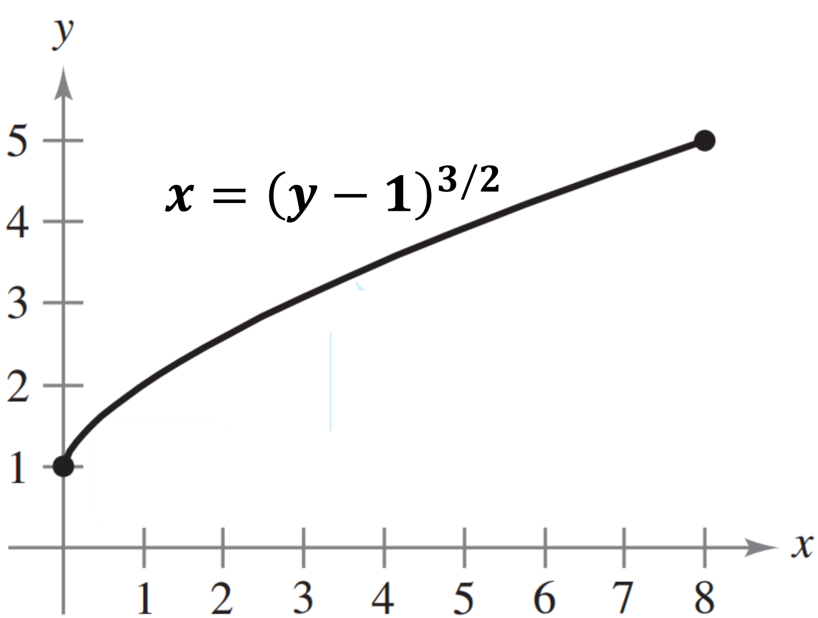
Arc Length with respect to Y

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

Where S is the length of the arc of $g(y)$ from $y = c$ to $y = d$.

$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Find the length of the arc of $x = (y-1)^{3/2}$ on the $x - interval$ $[0,8]$.



$$x = (y-1)^{3/2}$$

$$\frac{dx}{dy} = \frac{3}{2}(y-1)^{1/2}$$

$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_1^5 \sqrt{1 + \left[\frac{3}{2}(y-1)^{1/2}\right]^2}\, dy \approx 9.073$$

Homework

Section 7.4/8.4: Lengths of Curves

(1-17 odd, 21, 22)

Arc Length Homework

Section 7.4 (PDF)

($1 - 21$ every other odd, 23)

Find the distance between $(1, 3)$ and $(4,9)$ using the Distance Formula and also by using integration and the line $y = 2x + 1$