Optimization

Is the application of Minimum and Maximum values to terms such as greatest profit, least cost, least time, smallest/greatest size etc.

Example 1: Find two numbers whose sum are 20 and whose product is as large as possible.

First, think about what we are trying to optimize.
We want to optimize (maximize) the product.

Come up with an equation for the product.

$$P = xy$$

Second, we want to develop a secondary equation that allows us to reduce the primary equation to a single variable.

$$x + y = 20$$

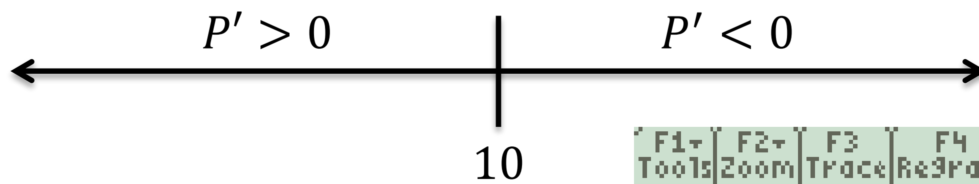
$$P(x) = x(20 - x)$$

$$P(x) = x(20 - x)$$

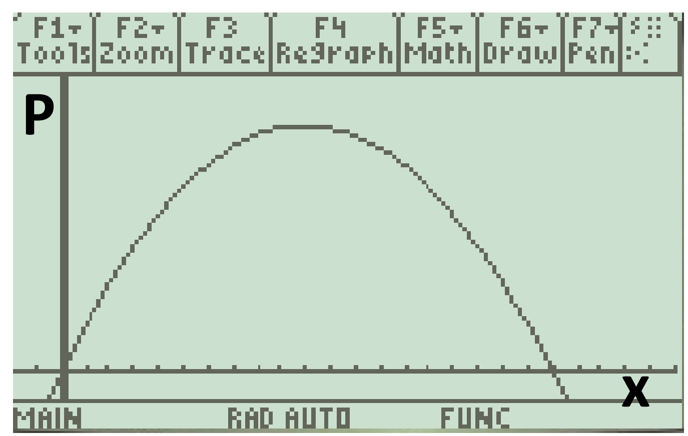
Now we want to use Calculus to determine our maximum.

$$P(x) = -x^2 + 20x \qquad P'(x) = -2x + 20$$

$$-2x + 20 = 0 \qquad x = 10$$



∴ we have a maximum at $x = 10$ and the product is 100.



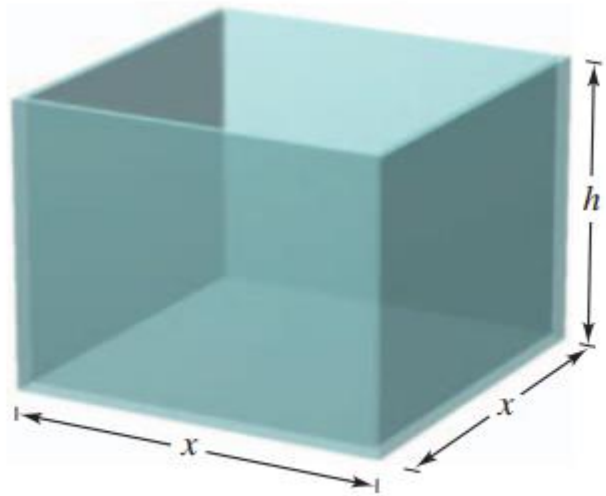
Keep in mind you can also use the Second Derivative Test to determine if it is a max/min

Example 2: A manufacturer wants to design an **open** box that has a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?

1. We want to draw a picture and label and develop a Primary Equation for what we are trying to optimize.

Maximize the volume

$$V = x^2 h$$



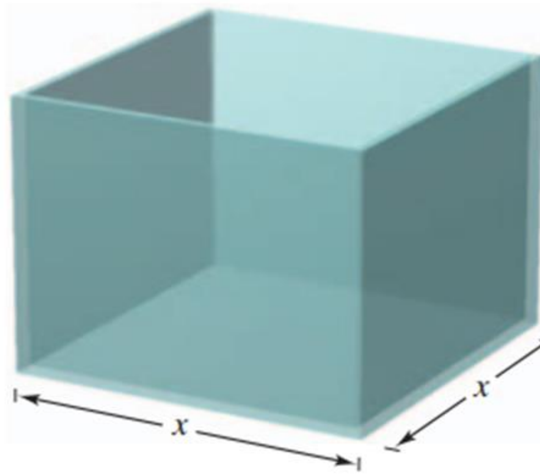
2. We want a secondary equation so we have only one independent variable.

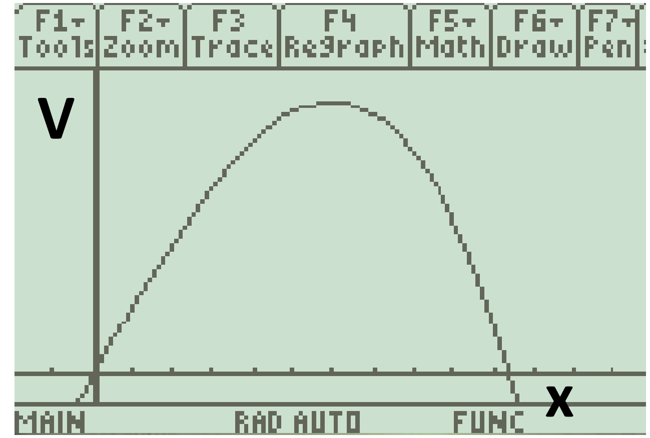
$$x^2 + 4xh = 108$$

$$V = x^2\left(\frac{108 - x^2}{4x}\right) = 27x - \frac{x^3}{4}$$

$$V = 27x - \frac{x^3}{4}$$

$$[0, \sqrt{108}]$$





3. We want to determine a domain for the function. What values of x make sense in this equation?

Since the surface area is 108 and x can't be negative $\quad 0 \leq x \leq \sqrt{108}$

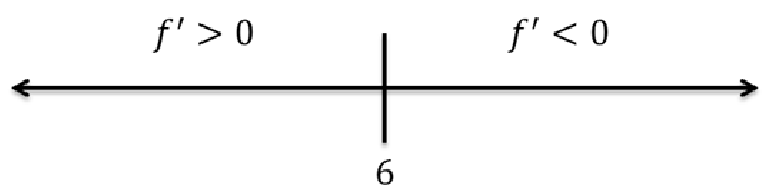
4. Find all critical numbers of the volume function and Verify Max/Min.

$$V' = 27 - \frac{3x^2}{4} = 0$$

$$3x^2 = 108$$

$$x = 6$$

~~$x = -6$~~



By the first derivative test we can see that the $x = 6$ is a maximum.
*Also keep in mind we should evaluate the endpoints in the original function (but we can logically see that would give a volume of 0).

Dimensions:
$6 \; x \; 6 \; x \; 3 \;\; inches$

Steps to solve an Optimization Problem.

1. Draw a Picture and write a **Primary Equation** for the quantity that is to be optimized (maximized or minimized)

2. Write a **secondary equation** in order to simplify first equation to a single independent variable

3. Determine the feasible domain. (Which values make sense)

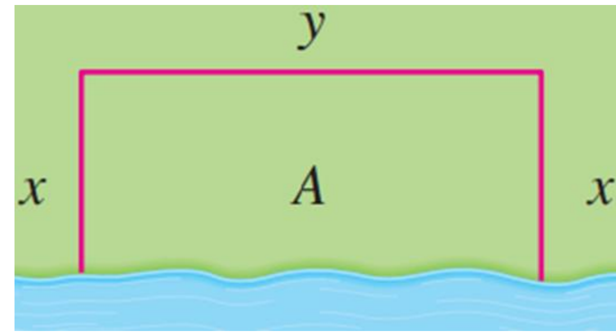
4. Use calculus techniques to find the Extrema (critical points or endpoints)

5. Verify the Extrema are a maximum or a minimum (First or Second Derivative Tests, Extreme Value Theorem, etc……)

A farmer has 2400 feet of fencing and wants to fence off a rectangular straight field that borders a straight river. What are the dimensions of the field that have the largest area?

1. Draw a picture/Primary Equation:

$$A = xy$$



2. Secondary Equation (One Independent Variable)

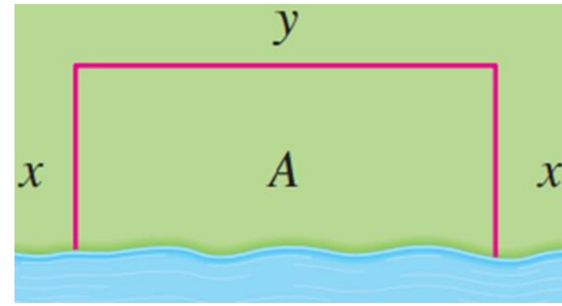
$$2x + y = 2400$$

$$A = x(2400 - 2x) = 2400x - 2x^2$$

3. Domain (What values of x make sense)

$$0 \leq x \leq 1200$$

A farmer has 2400 feet of fencing and wants to fence off a rectangular straight field that borders a straight river. What are the dimensions of the field that have the largest area?

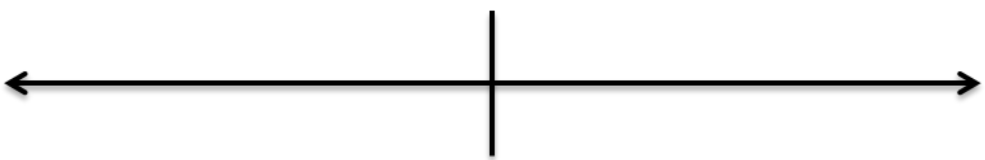


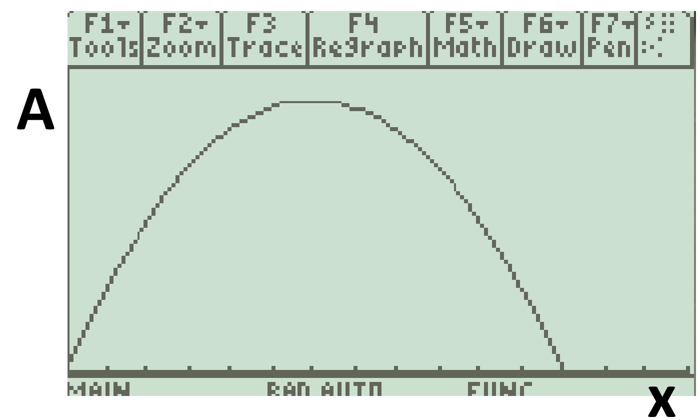
4. Take the derivative to find critical points and determine if they are maximums or minimums.

$$A(x) = 2400x - 2x^2$$

$$A'(x) = 2400 - 4x$$

$A'(x) = 0$ when $x = 600$

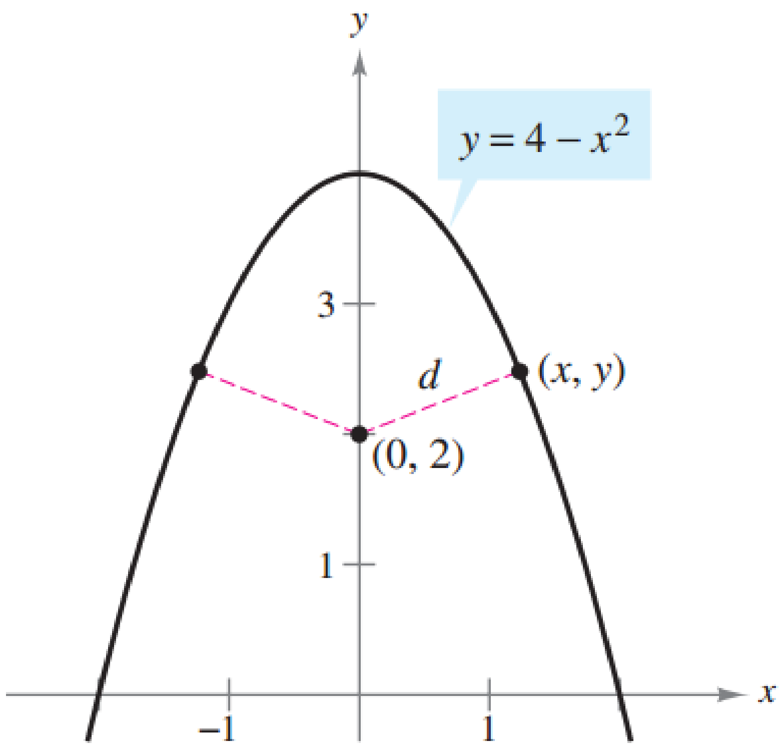




*Technically, we should also look at endpoint ($x = 0, x = 1200$), but we can logically eliminate these as both would give an area of 0.

We can thus see that Area is maximized when $x = 600$ and $y = 1200$ and equals 720,000.

Example 4: Which points on the graph of $y = 4 - x^2$ are closest to the point $(0,2)$?



Minimize Distance

1. $d = \sqrt{(x-0)^2 + (y-2)^2}$

2. $y = 4 - x^2$

$$d = \sqrt{(x-0)^2 + (4 - x^2 - 2)^2}$$

$$d = \sqrt{x^4 - 3x^2 + 4}$$

3. Determine the domain. The domain for this function is $(-\infty, \infty)$

4. Determine the minimum using Calculus.

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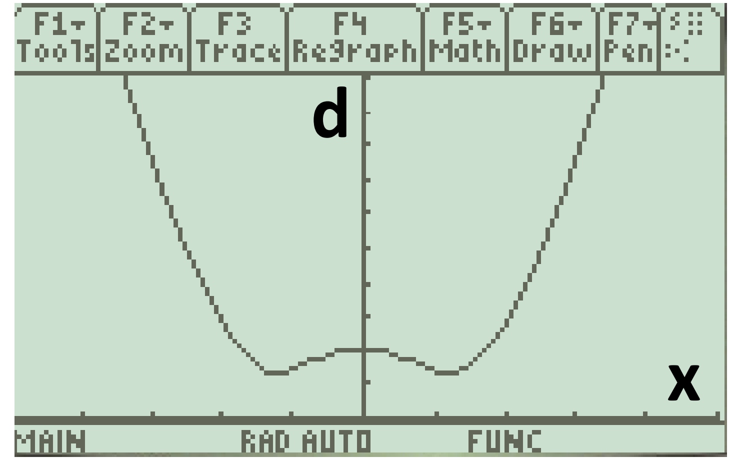
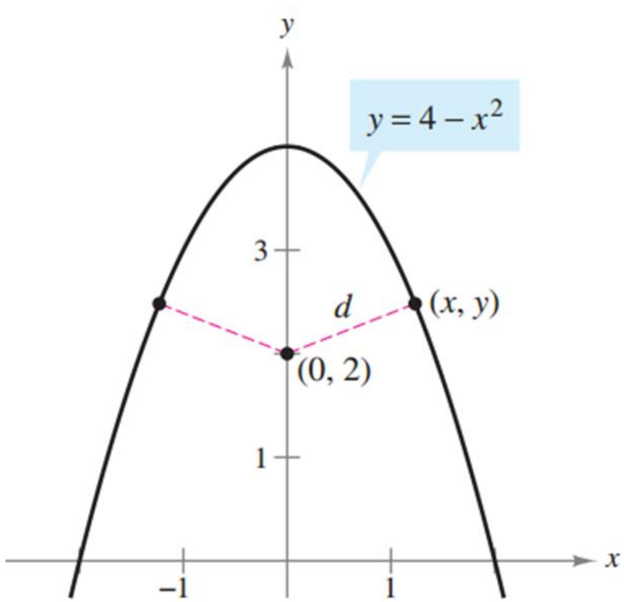


$$d = \sqrt{x^4 - 3x^2 + 4}$$

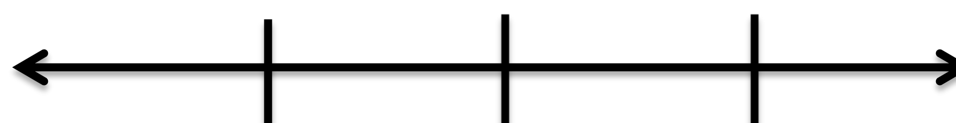
$$d' = \frac{4x^3 - 6x}{2\sqrt{x^4 - 3x^2 + 4}}$$

$$4x^3 - 6x = 0$$



$$2x(2x^2 - 3) = 0 \qquad x = 0, \sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}$$

*Can use 2nd or 1st Derivative Test



Both $x = \sqrt{\frac{3}{2}}$ and $-\sqrt{\frac{3}{2}}$ yield a minimum.

Minimum distance occurs at $\left(\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$ and $\left(-\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$

Homework

Section 3.7 P. 223

(4, 5, 19, 20, 22)

Homework

Section 3.7 P. 223

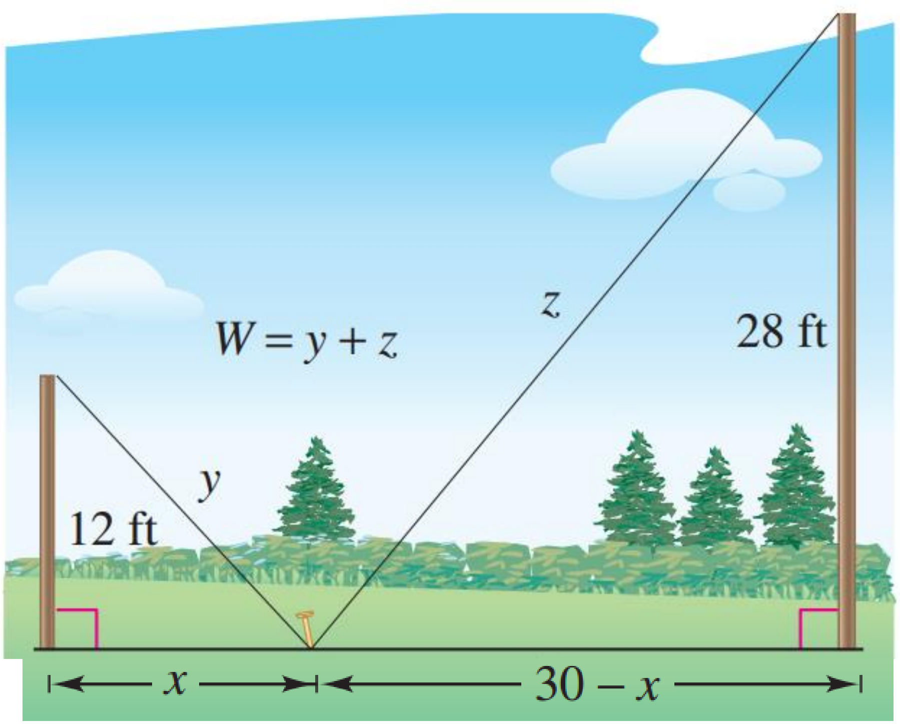
(5, 13, 15, 19)

(27, 33)

(23, 25, 29, 49)

Finding Minimum Length

Two posts, one 12 feet high and the other 28 feet high, stand 30 feet apart. They are to be stayed by two wires, attached to a single stake, running from ground level to the top of each post. Where should the stake be placed to use the least amount of wire?

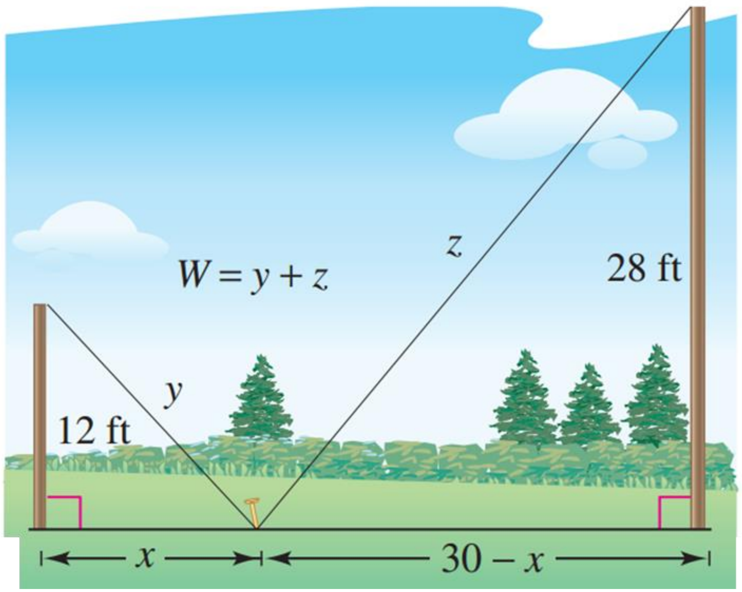


$$W = y + z$$

$$x^2 + 12^2 = y^2$$

$$(30 - x)^2 + 28^2 = z^2$$

$$W = \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684}$$



$$W = \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684}$$

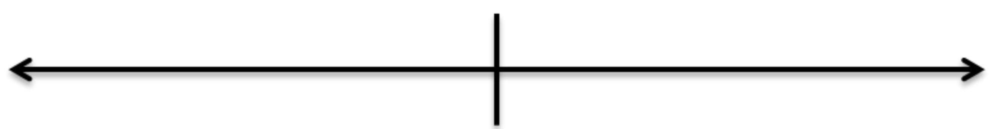
Domain: $0 \leq x \leq 30$

Keep in mind our goal, we are the looking for the $x - value$ that will minimize W.

$$\frac{dW}{dx} = \frac{x}{\sqrt{x^2 + 144}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1684}} = 0$$

$x = 9$, ~~$x = -22.5$~~

(Work above shown on next page)



$\boldsymbol{x = 9}$

The length of the wire, W, is thus minimized when $x = 9$.

Add a graph of the actual optimization function…

$$\frac{x}{\sqrt{x^2 + 144}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1684}} = 0$$

$$x\sqrt{x^2 - 60x + 1684} = (30 - x)\sqrt{x^2 + 144}$$

$$x^2(x^2 - 60x + 1684) = (30 - x)^2(x^2 + 144)$$

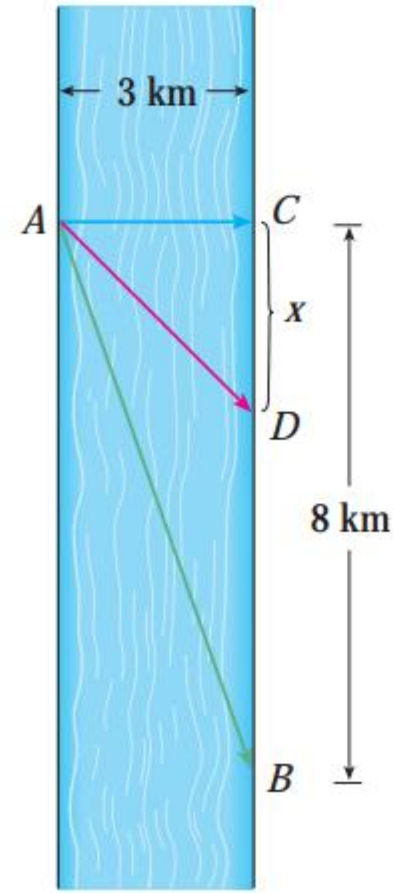
$$x^4 - 60x^3 + 1684x^2 = x^4 - 60x^3 + 1044x^2 - 8640x + 129{,}600$$

$$640x^2 + 8640x - 129{,}600 = 0$$

$$320(x - 9)(2x + 45) = 0$$

$$x = 9, -22.5$$

A man launches a boat from Point A on a bank of a straight river, 3km wide, and wants to reach point B, 8km downstream on the opposite bank as quickly as possible. If he can row at a rate of 6km/h and run at a rate of 8km/h, where should he land to reach point B as soon as possible?



$$Time = Rowing\ Time + Running\ Time$$

Minimize

$$\text{Time} = \frac{Distance}{Rate}$$

$$Rowing\ Time = \frac{\sqrt{3^2 + x^2}}{6} \qquad Running\ Time = \frac{8 - x}{8}$$

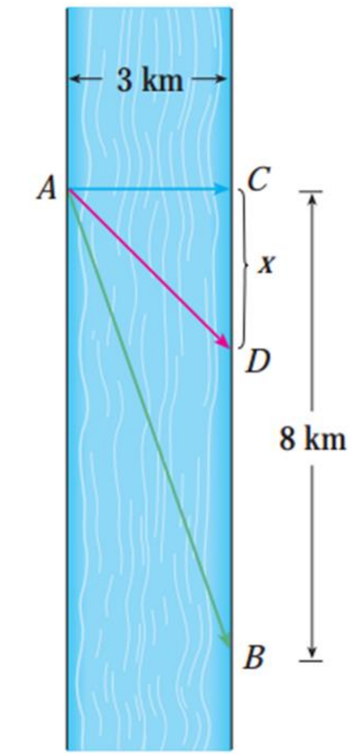
$$Time = T(x) = \frac{\sqrt{9 + x^2}}{6} + \frac{8 - x}{8}$$

The domain of the function will be $[0, 8]$

$$T'(x) = \frac{x}{6\sqrt{x^2 + 9}} - \frac{1}{8} \quad = 0$$

$$x = \frac{9}{\sqrt{7}}$$

Since we have a continuous closed function, we know there will be some minimum if we evaluate all critical points and endpoints in the original function.

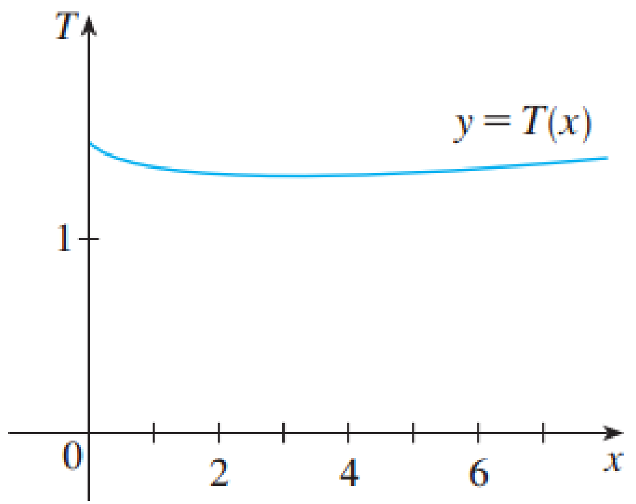


$$Critical\ Point:\ \ x = \frac{9}{\sqrt{7}}$$

$$Endpoint:\ \ x = 0, \qquad x = 8$$

$$T(x) = \frac{\sqrt{9 + x^2}}{6} + \frac{8 - x}{8}$$

$$T(0) = 1.5 \qquad T\left(\frac{9}{\sqrt{7}}\right) \approx 1.33 \qquad T(8) \approx 1.42$$



We can thus see that our minimum time will occur when $x = \frac{9}{\sqrt{7}}$.

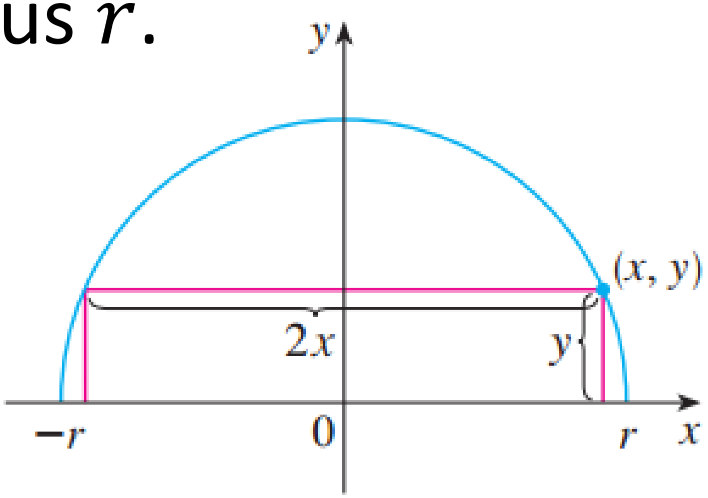
(The maximum time will occur if the person rows straight across the river and runs 8km).

Homework

3.7 P. 223 (23, 25, 29, 49)

29- See example 3

Find the rectangle with the largest area that can be inscribed in a semicircle of radius r.



Let the semicircle be the upper half of the circle $x^2 + y^2 = r^2$.

$$A = 2xy$$

Since we know that $x^2 + y^2 = r^2$, we can rearrange and get $y = \sqrt{r^2 - x^2}$

$$A = 2x\sqrt{r^2 - x^2}$$

Domain: $0 \le x \le r$

*Note r is a constant.

$$A' = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}} = 0$$

$$r^2 - 2x^2 = 0 \qquad r^2 = 2x^2 \qquad x = \sqrt{\frac{r^2}{2}}$$

$$A(0) = 0 \qquad A(r) = 0$$

$$A\left(\frac{r}{\sqrt{2}}\right) = r^2$$

Add a graph of the actual optimization function…

An open box is to be found by cutting equal squares from the corner of a 24 cm square piece of metal. Find the dimensions of the box that produce a maximum volume.

Add a graph of the actual optimization function…

Homework

3.7

P. 223 (27, 33, 40, 54)

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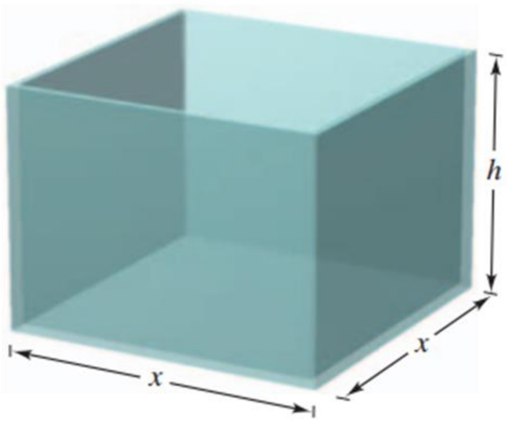
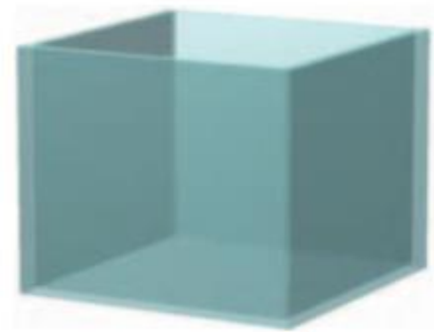
Example 2: A manufacturer wants to design an open box that has a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?



$$x^2 + 4xh = 108$$

Volume = $103\frac{3}{4}$



$5 \times 5 \times 4\frac{3}{20}$

Volume = 108

$6 \times 6 \times 3$

Volume = 88

$8 \times 8 \times 1\frac{3}{8}$

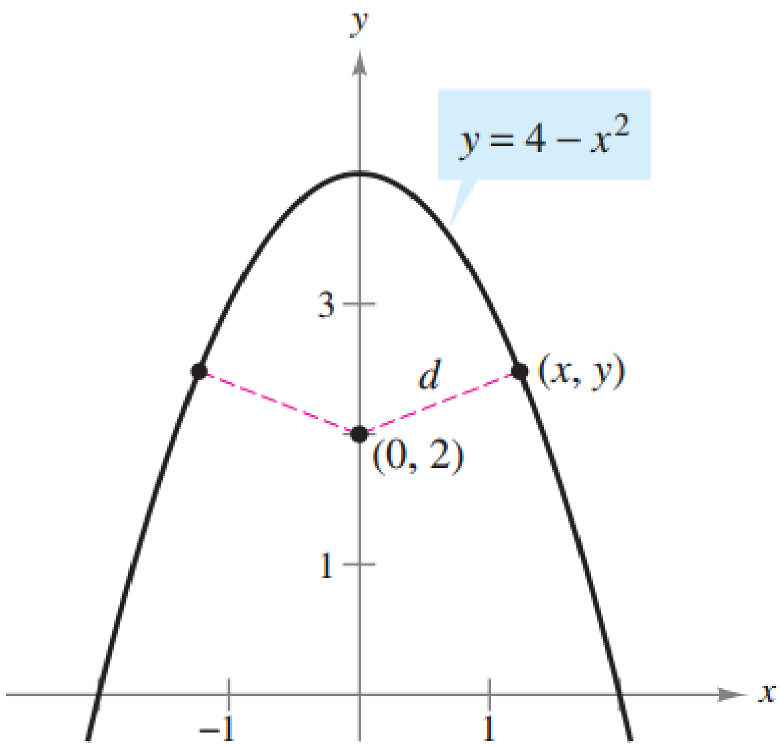
Volume = 92

$4 \times 4 \times 5\frac{3}{4}$

Volume = $74\frac{1}{4}$

$3 \times 3 \times 8\frac{1}{4}$

Example 4: Which points on the graph of $y = 4 - x^2$ are closest to the point $(0,2)$?



Minimize Distance

1. $d = \sqrt{(x-0)^2 + (y-2)^2}$

2. $y = 4 - x^2$

$$d = \sqrt{(x-0)^2 + (4 - x^2 - 2)^2}$$

$$d = \sqrt{x^4 - 3x^2 + 4}$$

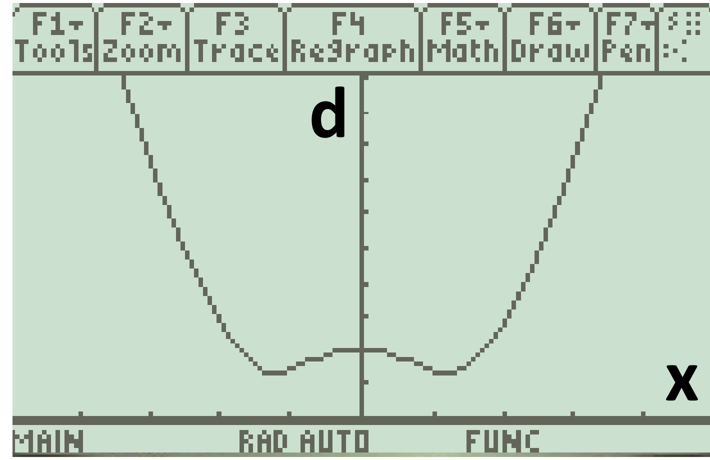
*Note that the x that minimizes $d = \sqrt{x^4 - 3x^2 + 4}$ will also minimize $d = x^4 - 3x^2 + 4$.

We want to minimize $d = x^4 - 3x^2 + 4$

3. Determine the domain.

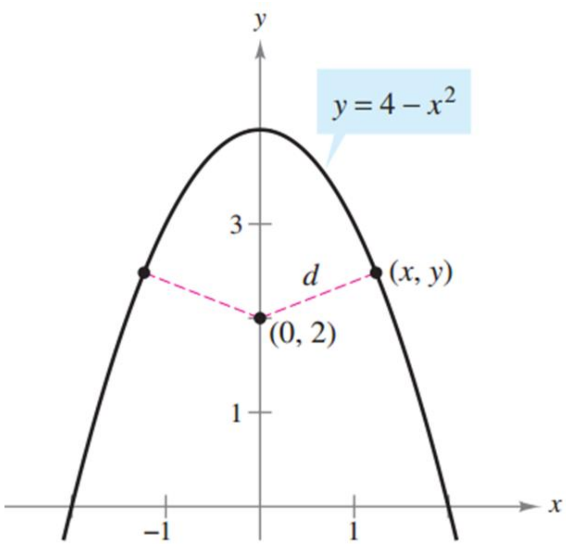
The domain for this function is $(-\infty, \infty)$

4. Determine the minimum using Calculus.



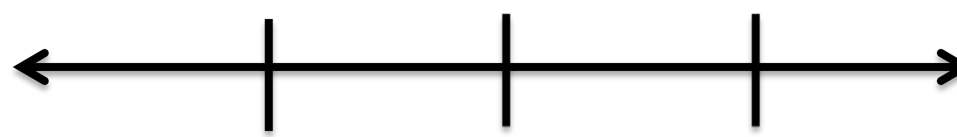
*Note that the x that minimizes $d = \sqrt{x^4 - 3x^2 + 4}$ will also minimize $d = x^4 - 3x^2 + 4.$

$$d' = 4x^3 - 6x$$



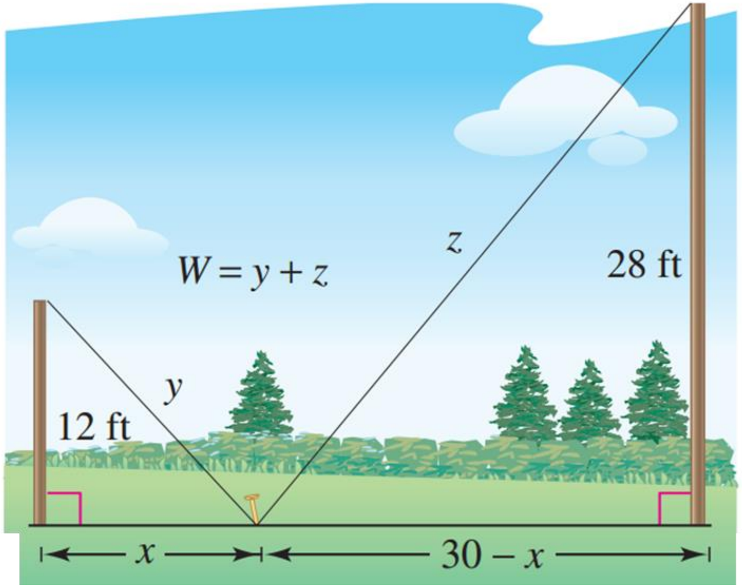
$$2x(2x^2 - 3) = 0$$

$$x = 0, \sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}$$



Both $x = \sqrt{\frac{3}{2}}$ and $-\sqrt{\frac{3}{2}}$ yield a minimum.

Minimum distance occurs at $\left(\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$ and $\left(-\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$



$$W = \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684}$$

Domain: $0 \leq x \leq 30$

Keep in mind our goal, we are the looking for the $x - value$ that will minimize W.

$$\frac{dW}{dx} = \frac{x}{\sqrt{x^2 + 144}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1684}} = 0 \qquad x = 9, \qquad \cancel{x = -22.5}$$

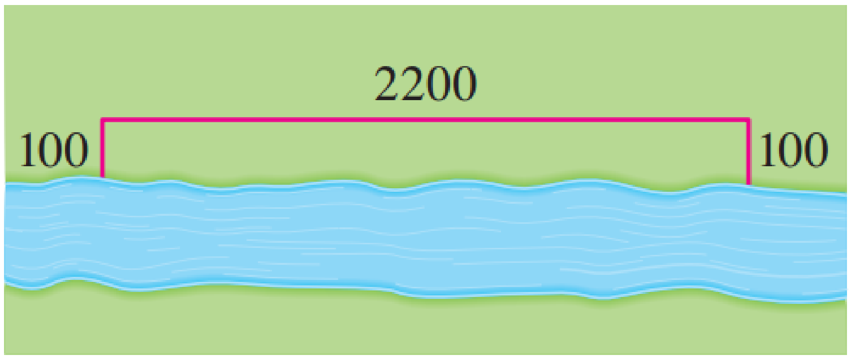
(Work above shown on next page)

We can use the extreme value theorem as we know there must be both a Maximum and a Minimum on a continuous closed interval.

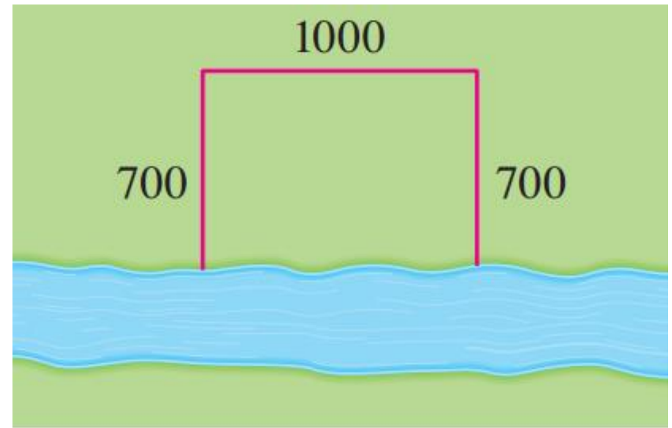
$W(0) \approx 53.04$ $\qquad$ $W(9) \approx 50$ $\qquad$ $W(30) \approx 60.31$

Therefore the length is minimized when $x = 9$ (stake placed 9 feet from the 12 foot pole).

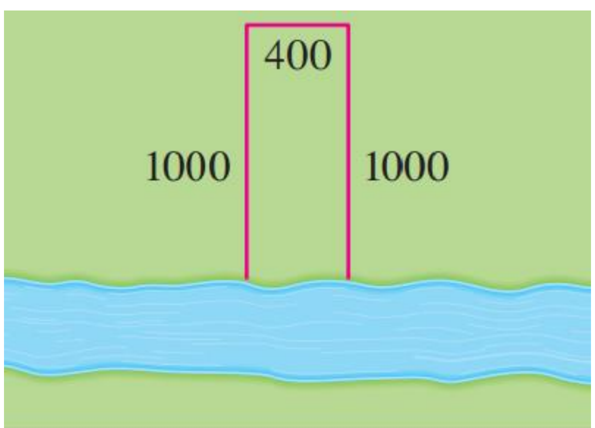
A farmer has 2400 feet of fencing and wants to fence off a rectangular straight field that borders a straight river. What are the dimensions of the field that have the largest area?



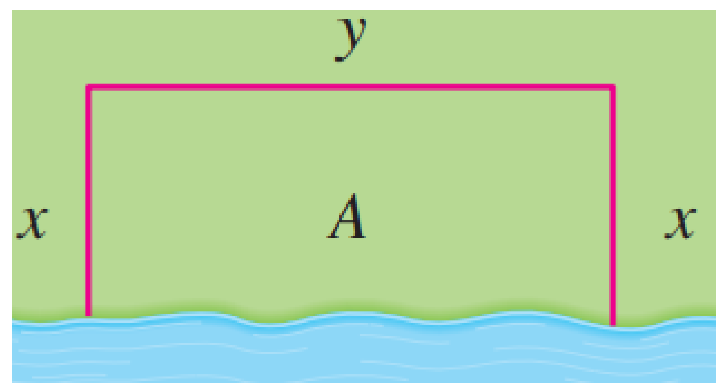
Area $= 100 \cdot 2200 = 220{,}000$ ft^2



Area $= 700 \cdot 1000 = 700{,}000$ ft^2

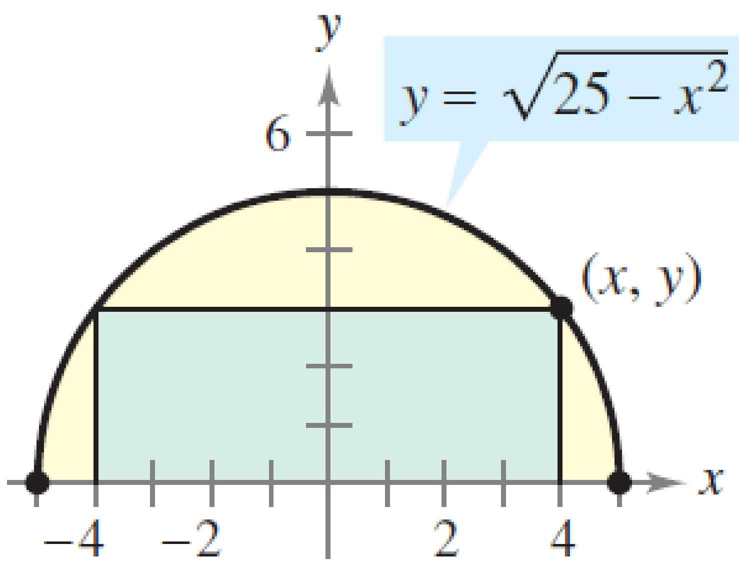


Area $= 1000 \cdot 400 = 400{,}000$ ft^2



$$A = xy$$

27. *Maximum Area* A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{25 - x^2}$ (see figure). What length and width should the rectangle have so that its area is a maximum?



Find the point on the function $f(x) = \sqrt{x-8}$ closest to the given point $(2,0)$:

Minimum Time A man is in a boat 2 miles from the nearest point on the coast. He is to go to a point Q, located 3 miles down the coast and 1 mile inland (see figure). He can row at 2 miles per hour and walk at 4 miles per hour. Toward what point on the coast should he row in order to reach point Q in the least time?