

# CHAPTER 9

## Infinite Series

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# CHAPTER 9

## Infinite Series

### Section 9.1 Sequences

1.  $a_n = 2^n$

$$a_1 = 2^1 = 2$$

$$a_2 = 2^2 = 4$$

$$a_3 = 2^3 = 8$$

$$a_4 = 2^4 = 16$$

$$a_5 = 2^5 = 32$$

2.  $a_n = \frac{3^n}{n!}$

$$a_1 = \frac{3}{1!} = 3$$

$$a_2 = \frac{3^2}{2!} = \frac{9}{2}$$

$$a_3 = \frac{3^3}{3!} = \frac{27}{6} = \frac{9}{2}$$

$$a_4 = \frac{3^4}{4!} = \frac{81}{24} = \frac{27}{8}$$

$$a_5 = \frac{3^5}{5!} = \frac{243}{120} = \frac{81}{40}$$

3.  $a_n = \left(-\frac{1}{2}\right)^n$

$$a_1 = \left(-\frac{1}{2}\right)^1 = -\frac{1}{2}$$

$$a_2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$a_3 = \left(-\frac{1}{2}\right)^3 = -\frac{1}{8}$$

$$a_4 = \left(-\frac{1}{2}\right)^4 = \frac{1}{16}$$

$$a_5 = \left(-\frac{1}{2}\right)^5 = -\frac{1}{32}$$

4.  $a_n = \left(-\frac{2}{3}\right)^n$

$$a_1 = -\frac{2}{3}$$

$$a_2 = \frac{4}{9}$$

$$a_3 = -\frac{8}{27}$$

$$a_4 = \frac{16}{81}$$

$$a_5 = -\frac{32}{243}$$

5.  $a_n = \sin \frac{n\pi}{2}$

$$a_1 = \sin \frac{\pi}{2} = 1$$

$$a_2 = \sin \pi = 0$$

$$a_3 = \sin \frac{3\pi}{2} = -1$$

$$a_4 = \sin 2\pi = 0$$

$$a_5 = \sin \frac{5\pi}{2} = 1$$

6.  $a_n = \frac{2n}{n+3}$

$$a_1 = \frac{2}{4} = \frac{1}{2}$$

$$a_2 = \frac{4}{5}$$

$$a_3 = \frac{6}{6} = 1$$

$$a_4 = \frac{8}{7}$$

$$a_5 = \frac{10}{8} = \frac{5}{4}$$

7.  $a_n = \frac{(-1)^{n(n+1)/2}}{n^2}$

$$a_1 = \frac{(-1)^1}{1^2} = -1$$

$$a_2 = \frac{(-1)^3}{2^2} = -\frac{1}{4}$$

$$a_3 = \frac{(-1)^6}{3^2} = \frac{1}{9}$$

$$a_4 = \frac{(-1)^{10}}{4^2} = \frac{1}{16}$$

$$a_5 = \frac{(-1)^{15}}{5^2} = -\frac{1}{25}$$

8.  $a_n = (-1)^{n+1} \left(\frac{2}{n}\right)$

$$a_1 = \frac{2}{1} = 2$$

$$a_2 = -\frac{2}{2} = -1$$

$$a_3 = \frac{2}{3}$$

$$a_4 = -\frac{2}{4} = -\frac{1}{2}$$

$$a_5 = \frac{2}{5}$$

9.  $a_n = 5 - \frac{1}{n} + \frac{1}{n^2}$

$$a_1 = 5 - 1 + 1 = 5$$

$$a_2 = 5 - \frac{1}{2} + \frac{1}{4} = \frac{19}{4}$$

$$a_3 = 5 - \frac{1}{3} + \frac{1}{9} = \frac{43}{9}$$

$$a_4 = 5 - \frac{1}{4} + \frac{1}{16} = \frac{77}{16}$$

$$a_5 = 5 - \frac{1}{5} + \frac{1}{25} = \frac{121}{25}$$

10.  $a_n = 10 + \frac{2}{n} + \frac{6}{n^2}$   
 $a_1 = 10 + 2 + 6 = 18$   
 $a_2 = 10 + 1 + \frac{3}{2} = \frac{25}{2}$   
 $a_3 = 10 + \frac{2}{3} + \frac{2}{3} = \frac{34}{3}$   
 $a_4 = 10 + \frac{1}{2} + \frac{3}{8} = \frac{87}{8}$   
 $a_5 = 10 + \frac{2}{5} + \frac{6}{25} = \frac{266}{25}$

11.  $a_1 = 3, a_{k+1} = 2(a_k - 1)$   
 $a_2 = 2(a_1 - 1) = 2(3 - 1) = 4$   
 $a_3 = 2(a_2 - 1) = 2(4 - 1) = 6$   
 $a_4 = 2(a_3 - 1) = 2(6 - 1) = 10$   
 $a_5 = 2(a_4 - 1) = 2(10 - 1) = 18$

12.  $a_1 = 4, a_{k+1} = \left(\frac{k+1}{2}\right)a_k$   
 $a_2 = \left(\frac{1+1}{2}\right)a_1 = 4$   
 $a_3 = \left(\frac{2+1}{2}\right)a_2 = 6$   
 $a_4 = \left(\frac{3+1}{2}\right)a_3 = 12$   
 $a_5 = \left(\frac{4+1}{2}\right)a_4 = 30$

13.  $a_1 = 32, a_{k+1} = \frac{1}{2}a_k$   
 $a_2 = \frac{1}{2}a_1 = \frac{1}{2}(32) = 16$   
 $a_3 = \frac{1}{2}a_2 = \frac{1}{2}(16) = 8$   
 $a_4 = \frac{1}{2}a_3 = \frac{1}{2}(8) = 4$   
 $a_5 = \frac{1}{2}a_4 = \frac{1}{2}(4) = 2$

14.  $a_1 = 6, a_{k+1} = \frac{1}{3}a_k^2$   
 $a_2 = \frac{1}{3}a_1^2 = \frac{1}{3}(6^2) = 12$   
 $a_3 = \frac{1}{3}a_2^2 = \frac{1}{3}(12^2) = 48$   
 $a_4 = \frac{1}{3}a_3^2 = \frac{1}{3}(48^2) = 768$   
 $a_5 = \frac{1}{3}a_4^2 = \frac{1}{3}(768^2) = 196,608$

15.  $a_n = \frac{8}{n+1}, a_1 = 4, a_2 = \frac{8}{3}$ ,  
 decreases to 0; matches (f).

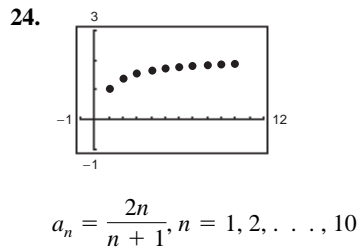
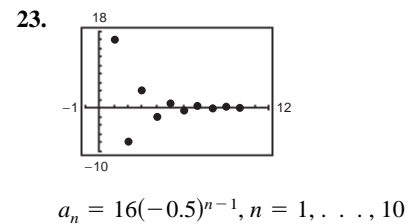
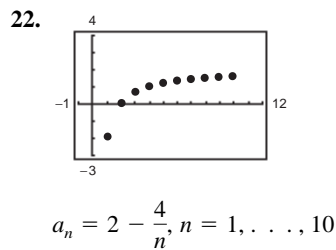
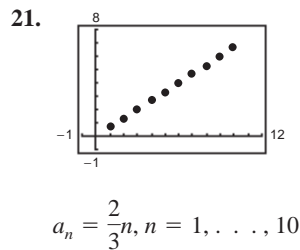
16.  $a_n = \frac{8n}{n+1}, a_1 = 4, a_2 = \frac{16}{3}$ ,  
 increases towards 8; matches (a).

17.  $a_n = 4(0.5)^{n-1}, a_1 = 4, a_2 = 2$ ,  
 decreases to 0; matches (e).

18.  $a_n = \frac{4^n}{n!}, a_1 = 4, a_2 = 8$ ,  
 eventually approaches 0; matches (b).

19.  $a_n(-1)^n, a_1 = -1, a_2 = 1, a_3 = -1$ , etc.; matches (d).

20.  $a_n = \frac{(-1)^n}{n}, a_1 = -1, a_2 = \frac{1}{2}, a_3 = -\frac{1}{3}$ , etc.; matches (c).



25.  $a_n = 3n - 1$   
 $a_5 = 3(5) - 1 = 14$   
 $a_6 = 3(6) - 1 = 17$   
 Add 3 to preceding term.

26.  $a_n = \frac{n+6}{2}$   
 $a_5 = \frac{5+6}{2} = \frac{11}{2}$   
 $a_6 = \frac{6+6}{2} = 6$

$$27. a_{n+1} = 2a_n, a_1 = 5$$

$$a_5 = 2(40) = 80$$

$$a_6 = 2(80) = 160$$

$$28. a_n = -\frac{1}{2}a_{n-1}, a_1 = 1$$

$$a_5 = \frac{1}{16}, a_6 = -\frac{1}{32}$$

$$29. a_n = \frac{3}{(-2)^{n-1}}$$

$$a_5 = \frac{3}{(-2)^4} = \frac{3}{16}$$

$$a_6 = \frac{3}{(-2)^5} = -\frac{3}{32}$$

Multiply the preceding term by  $-\frac{1}{2}$ .

$$30. a_n = -\frac{3}{2}a_{n-1}, a_1 = 1$$

$$a_5 = \frac{81}{16}, a_6 = -\frac{243}{32}$$

$$31. \frac{10!}{8!} = \frac{8!(9)(10)}{8!}$$

$$= (9)(10) = 90$$

$$32. \frac{25!}{23!} = \frac{23!(24)(25)}{23!}$$

$$= (24)(25) = 600$$

$$33. \frac{(n+1)!}{n!} = \frac{n!(n+1)}{n!} = n+1$$

$$34. \frac{(n+2)!}{n!} = \frac{n!(n+1)(n+2)}{n!}$$

$$= (n+1)(n+2)$$

$$35. \frac{(2n-1)!}{(2n+1)!} = \frac{(2n-1)!}{(2n-1)!(2n)(2n+1)}$$

$$= \frac{1}{2n(2n+1)}$$

$$36. \frac{(2n+2)!}{(2n)!} = \frac{(2n)!(2n+1)(2n+2)}{(2n)!}$$

$$= (2n+1)(2n+2)$$

$$37. \lim_{n \rightarrow \infty} \frac{5n^2}{n^2 + 2} = 5$$

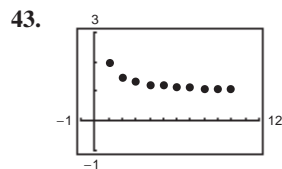
$$38. \lim_{n \rightarrow \infty} \left(5 - \frac{1}{n^2}\right) = 5 - 0 = 5$$

$$39. \lim_{n \rightarrow \infty} \frac{2n}{\sqrt{n^2 + 1}} = \lim_{n \rightarrow \infty} \frac{2}{\sqrt{1 + (1/n^2)}} = \frac{2}{1} = 2$$

$$40. \lim_{n \rightarrow \infty} \frac{5n}{\sqrt{n^2 + 4}} = \lim_{n \rightarrow \infty} \frac{5}{\sqrt{1 + (4/n^2)}} = \frac{5}{1} = 5$$

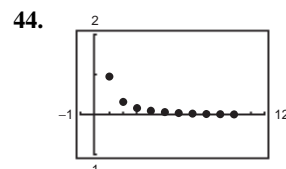
$$41. \lim_{n \rightarrow \infty} \sin\left(\frac{1}{n}\right) = 0$$

$$42. \lim_{n \rightarrow \infty} \cos\left(\frac{2}{n}\right) = 1$$



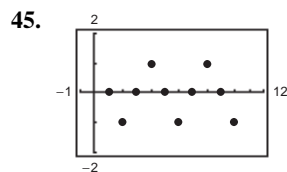
The graph seems to indicate that the sequence converges to 1. Analytically,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n+1}{n} = \lim_{x \rightarrow \infty} \frac{x+1}{x} = \lim_{x \rightarrow \infty} 1 = 1.$$



The graph seems to indicate that the sequence converges to 0. Analytically,

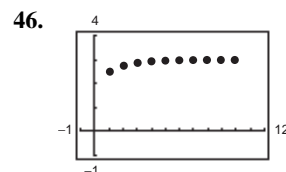
$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n^{3/2}} = \lim_{x \rightarrow \infty} \frac{1}{x^{3/2}} = 0.$$



The graph seems to indicate that the sequence diverges. Analytically, the sequence is

$$\{a_n\} = \{0, -1, 0, 1, 0, -1, \dots\}.$$

Hence,  $\lim_{n \rightarrow \infty} a_n$  does not exist.



The graph seems to indicate that the sequence converges to 3. Analytically,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(3 - \frac{1}{2^n}\right) = 3 - 0 = 3.$$

$$47. \lim_{n \rightarrow \infty} (-1)^n \left( \frac{n}{n+1} \right)$$

does not exist (oscillates between  $-1$  and  $1$ ), diverges.

$$48. \lim_{n \rightarrow \infty} [1 + (-1)^n]$$

does not exist, (alternates between  $0$  and  $2$ ), diverges.

$$49. \lim_{n \rightarrow \infty} \frac{3n^2 - n + 4}{2n^2 + 1} = \frac{3}{2}, \text{ converges}$$

$$50. \lim_{n \rightarrow \infty} \frac{\sqrt[3]{n}}{\sqrt[3]{n} + 1} = 1, \text{ converges}$$

$$51. a_n = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{(2n)^n}$$

$$= \frac{1}{2n} \cdot \frac{3}{2n} \cdot \frac{5}{2n} \cdot \dots \cdot \frac{2n-1}{2n} < \frac{1}{2n}$$

Thus,  $\lim_{n \rightarrow \infty} a_n = 0$ , converges.

52. The sequence diverges. To prove this analytically, we use mathematical induction to show that  $a_n \geq \left(\frac{3}{2}\right)^{n-1}$ . Clearly,  $a_1 = 1 \geq \left(\frac{3}{2}\right)^0 = 1$ . Assume that  $a_k \geq \left(\frac{3}{2}\right)^{k-1}$ . Then,

$$a_{k+1} = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k-1)(2k+1)}{(k+1)!} = a_k \frac{2k+1}{k+1} \geq \left(\frac{3}{2}\right)^{k-1} \left(\frac{2k+1}{k+1}\right).$$

Since

$$\frac{2k+1}{k+1} \geq \frac{3}{2},$$

we have  $a_{k+1} \geq \left(\frac{3}{2}\right)^k$ , which shows that  $a_n \geq \left(\frac{3}{2}\right)^{n-1}$  for all  $n$ .

$$53. \lim_{n \rightarrow \infty} \frac{1 + (-1)^n}{n} = 0, \text{ converges}$$

$$54. \lim_{n \rightarrow \infty} \frac{1 + (-1)^n}{n^2} = 0, \text{ converges}$$

$$55. \lim_{n \rightarrow \infty} \frac{\ln(n^3)}{2n} = \lim_{n \rightarrow \infty} \frac{3 \ln(n)}{2n}$$

$$= \lim_{n \rightarrow \infty} \frac{3 \left(\frac{1}{n}\right)}{2} = 0, \text{ converges}$$

(L'Hôpital's Rule)

$$56. \lim_{n \rightarrow \infty} \frac{\ln \sqrt{n}}{n} = \lim_{n \rightarrow \infty} \frac{1/2 \ln n}{n}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{2n} = 0, \text{ converges}$$

(L'Hôpital's Rule)

$$57. \lim_{n \rightarrow \infty} \left(\frac{3}{4}\right)^n = 0, \text{ converges}$$

$$58. \lim_{n \rightarrow \infty} (0.5)^n = 0, \text{ converges}$$

$$59. \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} = \lim_{n \rightarrow \infty} (n+1)$$

$$= \infty, \text{ diverges}$$

$$60. \lim_{n \rightarrow \infty} \frac{(n-2)!}{n!} = \lim_{n \rightarrow \infty} \frac{1}{n(n-1)} = 0, \text{ converges}$$

$$61. \lim_{n \rightarrow \infty} \left( \frac{n-1}{n} - \frac{n}{n-1} \right) = \lim_{n \rightarrow \infty} \frac{(n-1)^2 - n^2}{n(n-1)}$$

$$= \lim_{n \rightarrow \infty} \frac{1-2n}{n^2-n} = 0, \text{ converges}$$

$$62. \lim_{n \rightarrow \infty} \left( \frac{n^2}{2n+1} - \frac{n^2}{2n-1} \right) = \lim_{n \rightarrow \infty} \frac{-2n^2}{4n^2-1} = -\frac{1}{2},$$

converges

$$63. \lim_{n \rightarrow \infty} \frac{n^p}{e^n} = 0, \text{ converges}$$

( $p > 0, n \geq 2$ )

$$64. a_n = n \sin \frac{1}{n}$$

$$\text{Let } f(x) = x \sin \frac{1}{x}.$$

$$\lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} = \lim_{x \rightarrow \infty} \frac{(-1/x^2) \cos(1/x)}{-1/x^2} = \lim_{x \rightarrow \infty} \cos \frac{1}{x} = \cos 0 = 1 \quad (\text{L'Hôpital's Rule})$$

or,

$$\lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} = \lim_{y \rightarrow 0^+} \frac{\sin(y)}{y} = 1. \text{ Therefore } \lim_{n \rightarrow \infty} n \sin \frac{1}{n} = 1.$$

$$65. a_n = \left(1 + \frac{k}{n}\right)^n$$

$$66. \lim_{n \rightarrow \infty} 2^{1/n} = 2^0 = 1, \text{ converges}$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{k}{n}\right)^n = \lim_{u \rightarrow 0} [(1 + u)^{1/u}]^k = e^k$$

where  $u = k/n$ , converges

$$67. \lim_{n \rightarrow \infty} \frac{\sin n}{n} = \lim_{n \rightarrow \infty} (\sin n) \frac{1}{n} = 0,$$

$$68. \lim_{n \rightarrow \infty} \frac{\cos \pi n}{n^2} = 0, \text{ converges}$$

$$69. a_n = 3n - 2$$

converges (because  $(\sin n)$  is bounded)

$$70. a_n = 4n - 1$$

$$71. a_n = n^2 - 2$$

$$72. a_n = \frac{(-1)^{n-1}}{n^2}$$

$$73. a_n = \frac{n+1}{n+2}$$

$$74. a_n = \frac{(-1)^{n-1}}{2^{n-2}}$$

$$75. a_n = 1 + \frac{1}{n} = \frac{n+1}{n}$$

$$76. a_n = 1 + \frac{2^n - 1}{2^n} \\ = \frac{2^{n+1} - 1}{2^n}$$

$$77. a_n = \frac{n}{(n+1)(n+2)}$$

$$78. a_n = \frac{1}{n!}$$

$$79. a_n = \frac{(-1)^{n-1}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} \\ = \frac{(-1)^{n-1} 2^n n!}{(2n)!}$$

$$80. a_n = \frac{x^{n-1}}{(n-1)!}$$

$$81. a_n = (2n)!, n = 1, 2, 3, \dots$$

$$82. a_n = (2n-1)!, n = 1, 2, 3, \dots$$

$$83. a_n = 4 - \frac{1}{n} < 4 - \frac{1}{n+1} = a_{n+1},$$

monotonic;  $|a_n| < 4$ , bounded

$$84. \text{ Let } f(x) = \frac{3x}{x+2}. \text{ Then } f'(x) = \frac{6}{(x+2)^2}.$$

Thus,  $f$  is increasing which implies  $\{a_n\}$  is increasing.

$|a_n| < 3$ , bounded

$$85. \frac{n}{2^{n+2}} \stackrel{?}{\geq} \frac{n+1}{2^{(n+1)+2}}$$

$$2^{n+3}n \stackrel{?}{\geq} 2^{n+2}(n+1)$$

$$2n \stackrel{?}{\geq} n+1$$

$$n \geq 1$$

Hence,  $n \geq 1$

$$2n \geq n+1$$

$$2^{n+3}n \geq 2^{n+2}(n+1)$$

$$\frac{n}{2^{n+2}} \geq \frac{n+1}{2^{(n+1)+2}}$$

$$a_n \geq a_{n+1}$$

True; monotonic;  $|a_n| \leq \frac{1}{8}$ , bounded

$$87. a_n = (-1)^n \left(\frac{1}{n}\right)$$

$$a_1 = -1$$

$$a_2 = \frac{1}{2}$$

$$a_3 = -\frac{1}{3}$$

Not monotonic;  $|a_n| \leq 1$ , bounded

$$90. a_n = \left(\frac{3}{2}\right)^n < \left(\frac{3}{2}\right)^{n+1} = a_{n+1}$$

Monotonic;  $\lim_{n \rightarrow \infty} a_n = \infty$ , not bounded

$$93. a_n = \frac{\cos n}{n}$$

$$a_1 = 0.5403$$

$$a_2 = -0.2081$$

$$a_3 = -0.3230$$

$$a_4 = -0.1634$$

Not monotonic;  $|a_n| \leq 1$ , bounded

$$86. a_n = ne^{-n/2}$$

$$a_1 = 0.6065$$

$$a_2 = 0.7358$$

$$a_3 = 0.6694$$

Not monotonic;  $|a_n| \leq 0.7358$ , bounded

$$88. a_n = \left(-\frac{2}{3}\right)^n$$

$$a_1 = -\frac{2}{3}$$

$$a_2 = \frac{4}{9}$$

$$a_3 = -\frac{8}{27}$$

Not monotonic;  $|a_n| \leq \frac{2}{3}$ , bounded

$$91. a_n = \sin\left(\frac{n\pi}{6}\right)$$

$$a_1 = 0.500$$

$$a_2 = 0.8660$$

$$a_3 = 1.000$$

$$a_4 = 0.8660$$

Not monotonic;  $|a_n| \leq 1$ , bounded

$$89. a_n = \left(\frac{2}{3}\right)^n > \left(\frac{2}{3}\right)^{n+1} = a_{n+1}$$

Monotonic;  $|a_n| \leq \frac{2}{3}$ , bounded

$$92. a_n = \cos\left(\frac{n\pi}{2}\right)$$

$$a_1 = \cos \frac{\pi}{2} = 0$$

$$a_2 = \cos \pi = -1$$

$$a_3 = \cos\left(\frac{3\pi}{2}\right) = 0$$

Not monotonic;  $|a_n| \leq 1$ , bounded

$$94. a_n = \frac{\sin \sqrt{n}}{n}$$

$$a_1 = \frac{\sin(1)}{1} \approx 0.8415$$

$$a_4 = \frac{\sin(4)}{4} \approx -0.1892$$

$$a_2 = \frac{\sin(2)}{2} \approx 0.4546$$

$$a_5 = \frac{\sin(5)}{5} \approx -0.1918$$

$$a_3 = \frac{\sin(3)}{3} \approx 0.0470$$

$$a_6 = \frac{\sin(6)}{6} \approx -0.0466$$

Not monotonic,  $|a_n| \leq 1$ , bounded

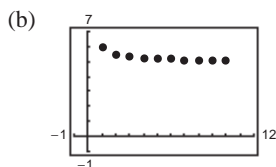
95. (a)  $a_n = 5 + \frac{1}{n}$

$$\left| 5 + \frac{1}{n} \right| \leq 6 \Rightarrow \{a_n\}, \text{ bounded}$$

$$a_n = 5 + \frac{1}{n} > 5 + \frac{1}{n+1}$$

$$= a_{n+1} \Rightarrow \{a_n\}, \text{ monotonic}$$

Therefore,  $\{a_n\}$  converges.



$$\lim_{n \rightarrow \infty} \left( 5 + \frac{1}{n} \right) = 5$$

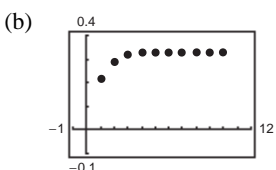
97. (a)  $a_n = \frac{1}{3} \left( 1 - \frac{1}{3^n} \right)$

$$\left| \frac{1}{3} \left( 1 - \frac{1}{3^n} \right) \right| < \frac{1}{3} \Rightarrow \{a_n\}, \text{ bounded}$$

$$a_n = \frac{1}{3} \left( 1 - \frac{1}{3^n} \right) < \frac{1}{3} \left( 1 - \frac{1}{3^{n+1}} \right)$$

$$= a_{n+1} \Rightarrow \{a_n\}, \text{ monotonic}$$

Therefore,  $\{a_n\}$  converges.



$$\lim_{n \rightarrow \infty} \left[ \frac{1}{3} \left( 1 - \frac{1}{3^n} \right) \right] = \frac{1}{3}$$

99.  $\{a_n\}$  has a limit because it is a bounded, monotonic sequence. The limit is less than or equal to 4, and greater than or equal to 2.

$$2 \leq \lim_{n \rightarrow \infty} a_n \leq 4$$

101.  $A_n = P \left( 1 + \frac{r}{12} \right)^n$

(a) No, the sequence  $\{A_n\}$  diverges,  $\lim_{n \rightarrow \infty} A_n = \infty$ .

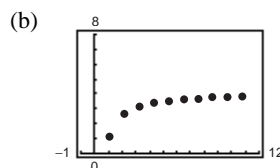
The amount will grow arbitrarily large over time.

96. (a)  $a_n = 4 - \frac{3}{n}$

$$\left| 4 - \frac{3}{n} \right| < 4 \Rightarrow \text{bounded}$$

$$a_n = 4 - \frac{3}{n} < 3 - \frac{4}{n+1} = a_{n+1} \Rightarrow \text{monotonic}$$

Therefore,  $\{a_n\}$  converges.



$$\lim_{n \rightarrow \infty} \left( 4 - \frac{3}{n} \right) = 4$$

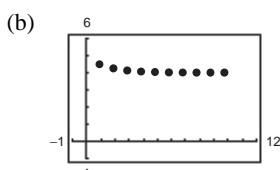
98. (a)  $a_n = 4 + \frac{1}{2^n}$

$$\left| 4 + \frac{1}{2^n} \right| \leq 4.5 \Rightarrow \{a_n\}, \text{ bounded}$$

$$a_n = 4 + \frac{1}{2^n} > 4 + \frac{1}{2^{n+1}}$$

$$= a_{n+1} \Rightarrow \{a_n\}, \text{ monotonic}$$

Therefore,  $\{a_n\}$  converges.



$$\lim_{n \rightarrow \infty} \left( 4 + \frac{1}{2^n} \right) = 4$$

100. The sequence  $\{a_n\}$  could converge or diverge. If  $\{a_n\}$  is increasing, then it converges to a limit less than or equal to 1. If  $\{a_n\}$  is decreasing, then it could converge (example:  $a_n = 1/n$ ) or diverge (example:  $a_n = -n$ ).

(b)  $P = 9000, r = 0.055, A_n = 9000 \left( 1 + \frac{0.055}{12} \right)^n$

$$A_1 = 9041.25 \quad A_6 = 9250.35$$

$$A_2 = 9082.69 \quad A_7 = 9292.75$$

$$A_3 = 9124.32 \quad A_8 = 9335.34$$

$$A_4 = 9166.14 \quad A_9 = 9378.13$$

$$A_5 = 9208.15 \quad A_{10} = 9421.11$$



102. (a)  $A_n = 100(401)(1.0025^n - 1)$

$$A_0 = 0$$

$$A_1 = 100.25$$

$$A_2 = 200.75$$

$$A_3 = 301.50$$

$$A_4 = 402.51$$

$$A_5 = 503.76$$

$$A_6 = 605.27$$

(b)  $A_{60} = 6480.83$

(c)  $A_{240} = 32,912.28$

 104. The first sequence because every other point is below the  $x$ -axis.

105.  $a_n = 10 - \frac{1}{n}$

106. Impossible. The sequence converges by Theorem 9.5.

107.  $a_n = \frac{3n}{4n+1}$

108. Impossible. An unbounded sequence diverges.

109. (a)  $A_n = (0.8)^n (2.5)$  billion

(b)  $A_1 = \$2$  billion

$$A_2 = \$1.6$$
 billion

$$A_3 = \$1.28$$
 billion

$$A_4 = \$1.024$$
 billion

(c)  $\lim_{n \rightarrow \infty} (0.8)^n (2.5) = 0$

110.  $P_n = 16,000(1.045)^n$

$$P_1 = \$16,720.00$$

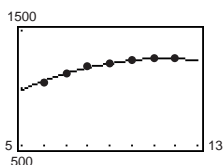
$$P_2 = \$17,472.40$$

$$P_3 \approx \$18,258.66$$

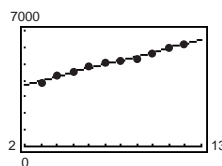
$$P_4 \approx \$19,080.30$$

$$P_5 \approx \$19,938.91$$

111. (a)  $a_n = -6.60n^2 + 151.7n + 387$


 (b) In 2008,  $n = 18$  and  $a_{18} \approx 979$  endangered species.

112. (a)  $a_n = 244.2n + 3250$


 (b) For 2008,  $n = 18$  and  $a_{18} \approx 7646$  million.

113.  $a_n = \frac{10^n}{n!}$

$$\begin{aligned} \text{(a) } a_9 &= a_{10} = \frac{10^9}{9!} \\ &= \frac{1,000,000,000}{362,880} \\ &= \frac{1,562,500}{567} \end{aligned}$$

(b) Decreasing

(c) Factorials increase more rapidly than exponentials.

114.  $a_n = \left(1 + \frac{1}{n}\right)^n$

$$a_1 = 2.0000$$

$$a_2 = 2.2500$$

$$a_3 \approx 2.3704$$

$$a_4 \approx 2.4414$$

$$a_5 \approx 2.4883$$

$$a_6 \approx 2.5216$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

115.  $\{a_n\} = \{\sqrt[n]{n}\} = \{n^{1/n}\}$

$$a_1 = 1^{1/1} = 1$$

$$a_2 = \sqrt{2} \approx 1.4142$$

$$a_3 = \sqrt[3]{3} \approx 1.4422$$

$$a_4 = \sqrt[4]{4} \approx 1.4142$$

$$a_5 = \sqrt[5]{5} \approx 1.3797$$

$$a_6 = \sqrt[6]{6} \approx 1.3480$$

Let  $y = \lim_{n \rightarrow \infty} n^{1/n}$ .

$$\ln y = \lim_{n \rightarrow \infty} \left( \frac{1}{n} \ln n \right) = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{1/n}{1} = 0$$

Since  $\ln y = 0$ , we have  $y = e^0 = 1$ . Therefore,

$$\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1.$$

117. True

118. True

116. Since

$$\lim_{n \rightarrow \infty} s_n = L > 0,$$

there exists for each  $\varepsilon > 0$ ,

an integer  $N$  such that  $|s_n - L| < \varepsilon$  for every  $n > N$ .

Let  $\varepsilon = L > 0$  and we have,

$$|s_n - L| < L, \quad -L < s_n - L < L, \quad \text{or } 0 < s_n < 2L$$

for each  $n > N$ .

119. True

120. True

121.  $a_{n+2} = a_n + a_{n+1}$

(a)  $a_1 = 1$

$$a_2 = 1$$

$$a_3 = 1 + 1 = 2$$

$$a_4 = 2 + 1 = 3$$

$$a_5 = 3 + 2 = 5$$

$$a_6 = 5 + 3 = 8$$

(b)  $b_n = \frac{a_{n+1}}{a_n}, n \geq 1$

$$b_1 = \frac{1}{1} = 1$$

$$b_2 = \frac{2}{1} = 2$$

$$b_3 = \frac{3}{2}$$

$$b_4 = \frac{5}{3}$$

$$b_5 = \frac{8}{5}$$

$$a_7 = 8 + 5 = 13$$

$$a_8 = 13 + 8 = 21$$

$$a_9 = 21 + 13 = 34$$

$$a_{10} = 34 + 21 = 55$$

$$a_{11} = 55 + 34 = 89$$

$$a_{12} = 89 + 55 = 144$$

$$b_6 = \frac{13}{8}$$

$$b_7 = \frac{21}{13}$$

$$b_8 = \frac{34}{21}$$

$$b_9 = \frac{55}{34}$$

$$b_{10} = \frac{89}{55}$$

(c)  $1 + \frac{1}{b_{n-1}} = 1 + \frac{1}{a_n/a_{n-1}}$

$$= 1 + \frac{a_{n-1}}{a_n}$$

$$= \frac{a_n + a_{n-1}}{a_n} = \frac{a_{n+1}}{a_n} = b_n$$

(d) If  $\lim_{n \rightarrow \infty} b_n = \rho$ , then  $\lim_{n \rightarrow \infty} \left( 1 + \frac{1}{b_{n-1}} \right) = \rho$ .

Since  $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} b_{n-1}$ , we have

$$1 + (1/\rho) = \rho.$$

$$\rho + 1 = \rho^2$$

$$0 = \rho^2 - \rho - 1$$

$$\rho = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Since  $a_n$ , and thus  $b_n$ , is positive,  $\rho = \frac{1 + \sqrt{5}}{2} \approx 1.6180$ .

122.  $x_0 = 1, x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, n = 1, 2, \dots$

$$x_1 = 1.5 \quad x_6 = 1.414214$$

$$x_2 = 1.41667 \quad x_7 = 1.414214$$

$$x_3 = 1.414216 \quad x_8 = 1.414114$$

$$x_4 = 1.414214 \quad x_9 = 1.414214$$

$$x_5 = 1.414214 \quad x_{10} = 1.414214$$

The limit of the sequence appears to be  $\sqrt{2}$ . In fact, this sequence is Newton's Method applied to  $f(x) = x^2 - 2$ .

$$\begin{aligned}
 123. \quad (a) \quad a_1 &= \sqrt{2} \approx 1.4142 & (b) \quad a_n &= \sqrt{2 + a_{n-1}}, \quad n \geq 2, a_1 = \sqrt{2} \\
 a_2 &= \sqrt{2 + \sqrt{2}} \approx 1.8478 \\
 a_3 &= \sqrt{2 + \sqrt{2 + \sqrt{2}}} \approx 1.9616 \\
 a_4 &= \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} \approx 1.9904 \\
 a_5 &= \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}} \approx 1.9976
 \end{aligned}$$

(c) We first use mathematical induction to show that  $a_n \leq 2$ ; clearly  $a_1 \leq 2$ . So assume  $a_k \leq 2$ . Then

$$\begin{aligned}
 a_k + 2 &\leq 4 \\
 \sqrt{a_k + 2} &\leq 2 \\
 a_{k+1} &\leq 2.
 \end{aligned}$$

Now we show that  $\{a_n\}$  is an increasing sequence. Since  $a_n \geq 0$  and  $a_n \leq 2$ ,

$$\begin{aligned}
 (a_n - 2)(a_n + 1) &\leq 0 \\
 a_n^2 - a_n - 2 &\leq 0 \\
 a_n^2 &\leq a_n + 2 \\
 a_n &\leq \sqrt{a_n + 2} \\
 a_n &\leq a_{n+1}.
 \end{aligned}$$

Since  $\{a_n\}$  is a bounding increasing sequence, it converges to some number  $L$ , by Theorem 9.5.

$$\begin{aligned}
 \lim_{n \rightarrow \infty} a_n = L &\Rightarrow \sqrt{2 + L} = L \Rightarrow 2 + L = L^2 \Rightarrow L^2 - L - 2 = 0 \\
 &\Rightarrow (L - 2)(L + 1) = 0 \Rightarrow L = 2 \quad (L \neq -1)
 \end{aligned}$$

$$\begin{aligned}
 124. \quad (a) \quad a_1 &= \sqrt{6} \approx 2.4495 & (b) \quad a_n &= \sqrt{6 + a_{n-1}}, \quad n \geq 2, a_1 = \sqrt{6} \\
 a_2 &= \sqrt{6 + \sqrt{6}} \approx 2.9068 \\
 a_3 &= \sqrt{6 + \sqrt{6 + \sqrt{6}}} \approx 2.9844 \\
 a_4 &= \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}} \approx 2.9974 \\
 a_5 &= \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}}} \approx 2.9996
 \end{aligned}$$

(c) We first use mathematical induction to show that  $a_n \leq 3$ ; clearly  $a_1 \leq 3$ . So assume  $a_k \leq 3$ . Then

$$\begin{aligned}
 6 + a_k &\leq 9 \\
 \sqrt{6 + a_k} &\leq 3 \\
 a_{k+1} &\leq 3.
 \end{aligned}$$

Now we show that  $\{a_n\}$  is an increasing sequence. Since  $a_n \geq 0$  and  $a_n \leq 3$ ,

$$\begin{aligned}
 (a_n - 3)(a_n + 2) &\leq 0 \\
 a_n^2 - a_n - 6 &\leq 0 \\
 a_n^2 &\leq a_n + 6 \\
 a_n &\leq \sqrt{a_n + 6} \\
 a_n &\leq a_{n+1}.
 \end{aligned}$$

Since  $\{a_n\}$  is a bounded increasing sequence, it converges to some number  $L$ :  $\lim_{n \rightarrow \infty} a_n = L$ . Thus,

$$\begin{aligned}
 \sqrt{6 + L} = L &\Rightarrow 6 + L = L^2 \Rightarrow L^2 - L - 6 = 0 \\
 &\Rightarrow (L - 3)(L + 2) = 0 \Rightarrow L = 3 \quad (L \neq -2)
 \end{aligned}$$

125. (a) We use mathematical induction to show that

$$a_n \leq \frac{1 + \sqrt{1 + 4k}}{2}.$$

[Note that if  $k = 2$ , then  $a_n \leq 3$ , and if  $k = 6$ , then  $a_n \leq 3$ .] Clearly,

$$a_1 = \sqrt{k} \leq \frac{\sqrt{1 + 4k}}{2} \leq \frac{1 + \sqrt{1 + 4k}}{2}.$$

Before proceeding to the induction step, note that

$$\begin{aligned} 2 + 2\sqrt{1 + 4k} + 4k &= 2 + 2\sqrt{1 + 4k} + 4k \\ \frac{1 + \sqrt{1 + 4k}}{2} + k &= \frac{1 + 2\sqrt{1 + 4k} + 1 + 4k}{4} \\ \frac{1 + \sqrt{1 + 4k}}{2} + k &= \left[ \frac{1 + \sqrt{1 + 4k}}{2} \right]^2 \\ \sqrt{\frac{1 + \sqrt{1 + 4k}}{2} + k} &= \frac{1 + \sqrt{1 + 4k}}{2}. \end{aligned}$$

So assume  $a_n \leq \frac{1 + \sqrt{1 + 4k}}{2}$ . Then

$$\begin{aligned} a_n + k &\leq \frac{1 + \sqrt{1 + 4k}}{2} + k \\ \sqrt{a_n + k} &\leq \sqrt{\frac{1 + \sqrt{1 + 4k}}{2} + k} \\ a_{n+1} &\leq \frac{1 + \sqrt{1 + 4k}}{2}. \end{aligned}$$

$\{a_n\}$  is increasing because

$$\left( a_n - \frac{1 + \sqrt{1 + 4k}}{2} \right) \left( a_n - \frac{1 - \sqrt{1 + 4k}}{2} \right) \leq 0$$

$$a_n^2 - a_n - k \leq 0$$

$$a_n^2 \leq a_n + k$$

$$a_n \leq \sqrt{a_n + k}$$

$$a_n \leq a_{n+1}.$$

(b) Since  $\{a_n\}$  is bounded and increasing, it has a limit  $L$ .

(c)  $\lim_{n \rightarrow \infty} a_n = L$  implies that

$$\begin{aligned} L &= \sqrt{k + L} \Rightarrow L^2 = k + L \\ &\Rightarrow L^2 - L - k = 0 \\ &\Rightarrow L = \frac{1 \pm \sqrt{1 + 4k}}{2}. \end{aligned}$$

$$\text{Since } L > 0, L = \frac{1 + \sqrt{1 + 4k}}{2}.$$

126. (a)  $a_0 = 10, b_0 = 3$

$$a_1 = \frac{a_0 + b_0}{2} = \frac{10 + 3}{2} = 6.5$$

$$b_1 = \sqrt{a_0 b_0} = \sqrt{10(3)} \approx 5.4772$$

$$a_2 = \frac{a_1 + b_1}{2} \approx 5.9886$$

$$b_2 = \sqrt{a_1 b_1} \approx 5.9667$$

$$a_3 = \frac{a_2 + b_2}{2} \approx 5.9777$$

$$b_3 = \sqrt{a_2 b_2} \approx 5.9777$$

$$a_4 = \frac{a_3 + b_3}{2} \approx 5.9777$$

$$b_4 = \sqrt{a_3 b_3} \approx 5.9777$$

$$a_5 = \frac{a_4 + b_4}{2} \approx 5.9777$$

$$b_5 = \sqrt{a_4 b_4} \approx 5.9777$$

The terms of  $\{a_n\}$  are decreasing, and those of  $\{b_n\}$  are increasing. They both seem to approach the same limit.

—CONTINUED—

## 126. —CONTINUED—

(b) For  $n = 0$ , we need to show  $a_0 > a_1 > b_1 > b_0$ . Since  $a_0 > b_0$ ,  $2a_0 > a_0 + b_0 \Rightarrow a_0 > \frac{a_0 + b_0}{2} = a_1$ .

Since  $(a_0 - b_0)^2 > 0$ ,

$$\begin{aligned} a_0^2 - 2a_0b_0 + b_0^2 > 0 &\Rightarrow a_0^2 + 2a_0b_0 + b_0^2 > 4a_0b_0 \\ &\Rightarrow (a_0 + b_0)^2 > 4a_0b_0 \Rightarrow a_0 + b_0 > 2\sqrt{a_0b_0} \Rightarrow a_1 > b_1. \end{aligned}$$

Since  $a_0 > b_0$ ,  $a_0b_0 > b_0^2 \Rightarrow \sqrt{a_0b_0} > b_0 \Rightarrow b_1 > b_0$ . Thus, we have shown that  $a_0 > a_1 > b_1 > b_0$ .

Now assume  $a_k > a_{k+1} > b_{k+1} > b_k$ . Since  $a_{k+1} > b_{k+1}$ ,

$$2a_{k+1} > a_{k+1} + b_{k+1} \Rightarrow a_{k+1} > \frac{a_{k+1} + b_{k+1}}{2} = a_{k+2}.$$

Since  $(a_{k+1} - b_{k+1})^2 > 0$ ,

$$\begin{aligned} a_{k+1}^2 - 2a_{k+1}b_{k+1} + b_{k+1}^2 > 0 &\Rightarrow a_{k+1}^2 + 2a_{k+1}b_{k+1} + b_{k+1}^2 > 4a_{k+1}b_{k+1} \\ &\Rightarrow (a_{k+1} + b_{k+1})^2 > 4a_{k+1}b_{k+1} \\ &\Rightarrow a_{k+1} + b_{k+1} > 2\sqrt{a_{k+1}b_{k+1}} \\ &\Rightarrow a_{k+2} > b_{k+2}. \end{aligned}$$

Since  $a_{k+1} > b_{k+1}$ ,  $a_{k+1}b_{k+1} > b_{k+1}^2 \Rightarrow \sqrt{a_{k+1}b_{k+1}} > b_{k+1} \Rightarrow b_{k+2} > b_{k+1}$ .

Thus, we have shown that  $a_{k+1} > a_{k+2} > b_{k+2} > b_{k+1}$ .

(c)  $\{a_n\}$  converges because it is decreasing and bounded below by 0.  $\{b_n\}$  converges because it is increasing and bounded above by  $a_0$ .

(d) Let  $\lim_{n \rightarrow \infty} a_n = A$  and  $\lim_{n \rightarrow \infty} b_n = B$ . Then  $A = \frac{A+B}{2} \Rightarrow A = B$ .

127. (a)  $f(x) = \sin x$ ,  $a_n = n \sin \frac{1}{n}$

$$f'(x) = \cos x, \quad f'(0) = 1$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} n \sin \frac{1}{n} = \lim_{n \rightarrow \infty} \frac{\sin(1/n)}{(1/n)} = 1 = f'(0)$$

(b)  $f'(0) = \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h}$

$$= \lim_{h \rightarrow 0^+} \frac{f(h)}{h}$$

$$= \lim_{n \rightarrow \infty} \frac{f(1/n)}{(1/n)}$$

$$= \lim_{n \rightarrow \infty} nf\left(\frac{1}{n}\right)$$

$$= \lim_{n \rightarrow \infty} a_n$$

128.  $a_n = nr^n$

(a)  $r = \frac{1}{2}$ :  $a_n = n\left(\frac{1}{2}\right)^n = \frac{n}{2^n} \rightarrow 0$

(b)  $r = 1$ :  $a_n = n$ , diverges

(c)  $r = \frac{3}{2}$ :  $a_n = n\left(\frac{3}{2}\right)^n$ , diverges

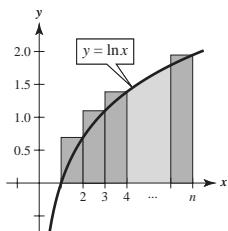
(d) If  $|r| \geq 1$ , diverges. If  $|r| < 1$ ,

$$nr^n = \frac{n}{r^{-n}} \quad \text{“}\frac{\infty}{\infty}\text{”}$$

$$\rightarrow \frac{1}{-r^{-n} \ln(r)} = \frac{-r^n}{\ln(r)} \rightarrow 0.$$

The sequence converges for  $|r| < 1$ .

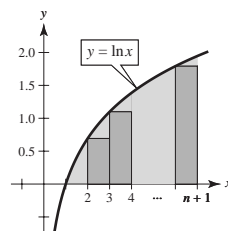
129. (a)



$$\int_1^n \ln x \, dx < \ln 2 + \ln 3 + \cdots + \ln n$$

$$= \ln[1 \cdot 2 \cdot 3 \cdots n] = \ln(n!)$$

(b)



$$\int_1^{n+1} \ln x \, dx > \ln 2 + \ln 3 + \cdots + \ln n$$

$$= \ln(n!)$$

$$(c) \int \ln x \, dx = x \ln x - x + C$$

$$\int_1^n \ln x \, dx = n \ln n - n + 1 = \ln n^n - n + 1$$

From part (a):  $\ln n^n - n + 1 < \ln(n!)$ 

$$e^{\ln n^n - n + 1} < n!$$

$$\frac{n^n}{e^{n-1}} < n!$$

$$\int_1^{n+1} \ln x \, dx = (n+1) \ln(n+1) - (n+1) + 1 = \ln(n+1)^{n+1} - n$$

From part (b):  $\ln(n+1)^{n+1} - n > \ln(n!)$ 

$$e^{\ln(n+1)^{n+1} - n} > n!$$

$$\frac{(n+1)^{n+1}}{e^n} > n!$$

$$(d) \frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}$$

$$\frac{n}{e^{1-(1/n)}} < \sqrt[n]{n!} < \frac{(n+1)^{(n+1)/n}}{e}$$

$$\frac{1}{e^{1-(1/n)}} < \frac{\sqrt[n]{n!}}{n} < \frac{(n+1)^{1+(1/n)}}{ne}$$

$$\lim_{n \rightarrow \infty} \frac{1}{e^{1-(1/n)}} = \frac{1}{e}$$

$$\lim_{n \rightarrow \infty} \frac{(n+1)^{1+(1/n)}}{ne} = \lim_{n \rightarrow \infty} \frac{(n+1)}{n} \frac{(n+1)^{1/n}}{e}$$

$$= (1) \frac{1}{e} = \frac{1}{e}$$

By the Squeeze Theorem,  $\lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$ .

$$(e) n = 20: \frac{\sqrt[20]{20!}}{20} \approx 0.4152$$

$$n = 50: \frac{\sqrt[50]{50!}}{50} \approx 0.3897$$

$$n = 100: \frac{\sqrt[100]{100!}}{100} \approx 0.3799$$

$$\frac{1}{e} \approx 0.3679$$

$$130. a_n = \frac{1}{n} \sum_{k=1}^n \frac{1}{1 + (k/n)}$$

$$(a) a_1 = \frac{1}{1+1} = \frac{1}{2} = 0.5$$

$$a_2 = \frac{1}{2} \left[ \frac{1}{1 + (1/2)} + \frac{1}{1+1} \right] = \frac{1}{2} \left[ \frac{2}{3} + \frac{1}{2} \right] = \frac{7}{12} \approx 0.5833$$

$$a_3 = \frac{1}{3} \left[ \frac{1}{1 + (1/3)} + \frac{1}{1 + (2/3)} + \frac{1}{1+1} \right] = \frac{37}{60} \approx 0.6167$$

$$a_4 = \frac{1}{4} \left[ \frac{1}{1 + (1/4)} + \frac{1}{1 + (2/4)} + \frac{1}{1 + (3/4)} + \frac{1}{1+1} \right] = \frac{533}{840} \approx 0.6345$$

$$a_5 = \frac{1}{5} \left[ \frac{1}{1 + (1/5)} + \frac{1}{1 + (2/5)} + \frac{1}{1 + (3/5)} + \frac{1}{1 + (4/5)} + \frac{1}{1+1} \right] = \frac{1627}{2520} \approx 0.6456$$

$$(b) \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{1 + (k/n)} = \int_0^1 \frac{1}{1+x} dx = \ln|1+x| \Big|_0^1 = \ln 2$$

131. For a given  $\epsilon > 0$ , we must find  $M > 0$  such that

$$|a_n - L| = \left| \frac{1}{n^3} \right| < \epsilon$$

whenever  $n > M$ . That is,

$$n^3 > \frac{1}{\epsilon} \text{ or } n > \left( \frac{1}{\epsilon} \right)^{1/3}.$$

So, let  $\epsilon > 0$  be given. Let  $M$  be an integer satisfying  $M > (1/\epsilon)^{1/3}$ . For  $n > M$ , we have

$$n > \left( \frac{1}{\epsilon} \right)^{1/3}$$

$$n^3 > \frac{1}{\epsilon}$$

$$\epsilon > \frac{1}{n^3} \Rightarrow \left| \frac{1}{n^3} - 0 \right| < \epsilon.$$

$$\text{Thus, } \lim_{n \rightarrow \infty} \frac{1}{n^3} = 0.$$

133. If  $\{a_n\}$  is bounded, monotonic and nonincreasing, then  $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \geq \dots$ . Then  $-a_1 \leq -a_2 \leq -a_3 \leq \dots \leq -a_n \leq \dots$  is a bounded, monotonic, nondecreasing sequence which converges by the first half of the theorem. Since  $\{-a_n\}$  converges, then so does  $\{a_n\}$ .

132. For a given  $\epsilon > 0$ , we must find  $M > 0$  such that

$$|a_n - L| = |r^n| \epsilon \text{ whenever } n > M. \text{ That is, } n \ln|r| < \ln(\epsilon) \text{ or}$$

$$n > \frac{\ln(\epsilon)}{\ln|r|} \text{ (since } \ln|r| < 0 \text{ for } |r| < 1).$$

So, let  $\epsilon > 0$  be given. Let  $M$  be an integer satisfying

$$M > \frac{\ln(\epsilon)}{\ln|r|}.$$

For  $n > M$ , we have

$$n > \frac{\ln(\epsilon)}{\ln|r|}$$

$$n \ln|r| < \ln(\epsilon)$$

$$\ln|r|^n < \ln(\epsilon)$$

$$|r|^n < \epsilon$$

$$|r^n - 0| < \epsilon.$$

Thus,  $\lim_{n \rightarrow \infty} r^n = 0$  for  $-1 < r < 1$ .

134. Define  $a_n = \frac{x_{n+1} + x_{n-1}}{x_n}$ ,  $n \geq 1$ .

$$x_{n+1}^2 - x_n x_{n+2} = 1 = x_n^2 - x_{n-1} x_{n+1} \Rightarrow$$

$$x_{n+1}(x_{n+1} + x_{n-1}) = x_n(x_n + x_{n+2})$$

$$\frac{x_{n+1} + x_{n-1}}{x_n} = \frac{x_{n+2} + x_n}{x_{n+1}}$$

$$a_n = a_{n+1}$$

Hence,  $a_1 = a_2 = \dots = a$ . Thus,

$$x_{n+1} = a_n x_n - x_{n-1} = a x_n - x_{n-1}.$$

135.  $T_n = n! + 2^n$

We use mathematical induction to verify the formula.

$$T_0 = 1 + 1 = 2$$

$$T_1 = 1 + 2 = 3$$

$$T_2 = 2 + 4 = 6$$

Assume  $T_k = k! + 2^k$ . Then

$$\begin{aligned} T_{k+1} &= (k+1+4)T_k - 4(k+1)T_{k-1} + (4(k+1)-8)T_{k-2} \\ &= (k+5)[k! + 2^k] - 4(k+1)((k-1)! + 2^{k-1}) + (4k-4)((k-2)! + 2^{k-2}) \\ &= [(k+5)(k)(k-1) - 4(k+1)(k-1) + 4(k-1)](k-2)! + [(k+5)4 - 8(k+1) + 4(k-1)]2^{k-2} \\ &= [k^2 + 5k - 4k - 4 + 4](k-1)! + 8 \cdot 2^{k-2} \\ &= (k+1)! + 2^{k+1}. \end{aligned}$$

By mathematical induction, the formula is valid for all  $n$ .

## Section 9.2 Series and Convergence

1.  $S_1 = 1$

$$S_2 = 1 + \frac{1}{4} = 1.2500$$

$$S_3 = 1 + \frac{1}{4} + \frac{1}{9} \approx 1.3611$$

$$S_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \approx 1.4236$$

$$S_5 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} \approx 1.4636$$

2.  $S_1 = \frac{1}{6} \approx 0.1667$

$$S_2 = \frac{1}{6} + \frac{1}{6} \approx 0.3333$$

$$S_3 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} \approx 0.4833$$

$$S_4 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{2}{15} \approx 0.6167$$

$$S_5 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{2}{15} + \frac{5}{42} \approx 0.7357$$

3.  $S_1 = 3$

$$S_2 = 3 - \frac{9}{2} = -1.5$$

$$S_3 = 3 - \frac{9}{2} + \frac{27}{4} = 5.25$$

$$S_4 = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} = -4.875$$

$$S_5 = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \frac{243}{16} = 10.3125$$

4.  $S_1 = 1$

$$S_2 = 1 + \frac{1}{3} \approx 1.3333$$

$$S_3 = 1 + \frac{1}{3} + \frac{1}{5} \approx 1.5333$$

$$S_4 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} \approx 1.6762$$

$$S_5 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} \approx 1.7873$$

5.  $S_1 = 3$

$$S_2 = 3 + \frac{3}{2} = 4.5$$

$$S_3 = 3 + \frac{3}{2} + \frac{3}{4} = 5.250$$

$$S_4 = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} = 5.625$$

$$S_5 = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} = 5.8125$$

6.  $S_1 = 1$

$$S_2 = 1 - \frac{1}{2} = 0.5$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{6} \approx 0.6667$$

$$S_4 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} \approx 0.6250$$

$$S_5 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} \approx 0.6333$$

7.  $\sum_{n=0}^{\infty} 3\left(\frac{3}{2}\right)^n$

Geometric series

$$r = \frac{3}{2} > 1$$

Diverges by Theorem 9.6

8.  $\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^n$

Geometric series

$$r = \frac{4}{3} > 1$$

Diverges by Theorem 9.6

9.  $\sum_{n=0}^{\infty} 1000(1.055)^n$

Geometric series

$$r = 1.055 > 1$$

Diverges by Theorem 9.6



10.  $\sum_{n=0}^{\infty} 2(-1.03)^n$

Geometric series

$|r| = 1.03 > 1$

Diverges by Theorem 9.6

11.  $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$

Diverges by Theorem 9.9

12.  $\sum_{n=1}^{\infty} \frac{n}{2n+3}$

$\lim_{n \rightarrow \infty} \frac{n}{2n+3} = \frac{1}{2} \neq 0$

Diverges by Theorem 9.9

13.  $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$

$\lim_{n \rightarrow \infty} \frac{n^2}{n^2+1} = 1 \neq 0$

Diverges by Theorem 9.9

14.  $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 \neq 0$

Diverges by Theorem 9.9

15.  $\sum_{n=1}^{\infty} \frac{2^n+1}{2^{n+1}}$

$\lim_{n \rightarrow \infty} \frac{2^n+1}{2^{n+1}} = \lim_{n \rightarrow \infty} \frac{1+2^{-n}}{2} = \frac{1}{2} \neq 0$

Diverges by Theorem 9.9

16.  $\sum_{n=1}^{\infty} \frac{n!}{2^n}$

$\lim_{n \rightarrow \infty} \frac{n!}{2^n} = \infty$

Diverges by Theorem 9.9

17.  $\sum_{n=0}^{\infty} \frac{9}{4} \left(\frac{1}{4}\right)^n = \frac{9}{4} \left[1 + \frac{1}{4} + \frac{1}{16} + \dots\right]$

$S_0 = \frac{9}{4}, S_1 = \frac{9}{4} \cdot \frac{5}{4} = \frac{45}{16}, S_2 = \frac{9}{4} \cdot \frac{21}{16} \approx 2.95, \dots$

Matches graph (c). Analytically, the series is geometric:

$\sum_{n=0}^{\infty} \left(\frac{9}{4}\right) \left(\frac{1}{4}\right)^n = \frac{9/4}{1-1/4} = \frac{9/4}{3/4} = 3$

18.  $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \frac{4}{9} + \dots$

$S_0 = 1, S_1 = \frac{5}{3}, S_2 \approx 2.11, \dots$

Matches graph (b). Analytically, the series is geometric:

$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1-2/3} = \frac{1}{1/3} = 3$

19.  $\sum_{n=0}^{\infty} \frac{15}{4} \left(-\frac{1}{4}\right)^n = \frac{15}{4} \left[1 - \frac{1}{4} + \frac{1}{16} - \dots\right]$

$S_0 = \frac{15}{4}, S_1 = \frac{45}{16}, S_2 \approx 3.05, \dots$

Matches graph (a). Analytically, the series is geometric:

$\sum_{n=0}^{\infty} \frac{15}{4} \left(-\frac{1}{4}\right)^n = \frac{15/4}{1-(-1/4)} = \frac{15/4}{5/4} = 3$

20.  $\sum_{n=0}^{\infty} \frac{17}{3} \left(-\frac{8}{9}\right)^n = \frac{17}{3} \left[1 - \frac{8}{9} + \frac{64}{81} - \dots\right]$

$S_0 = \frac{17}{3}, S_1 \approx 0.63, S_3 \approx 5.1, \dots$

Matches (d). Analytically, the series is geometric:

$\sum_{n=0}^{\infty} \frac{17}{3} \left(-\frac{8}{9}\right)^n = \frac{17/3}{1-(-8/9)} = \frac{17/3}{17/9} = 3$

21.  $\sum_{n=0}^{\infty} \frac{17}{3} \left(-\frac{1}{2}\right)^n = \frac{17}{3} \left[1 - \frac{1}{2} + \frac{1}{4} - \dots\right]$

$S_0 = \frac{17}{3}, S_1 = \frac{17}{6}, \dots$

Matches (f). The series is geometric:

$\sum_{n=0}^{\infty} \frac{17}{3} \left(-\frac{1}{2}\right)^n = \frac{17}{3} \frac{1}{1-(-1/2)} = \frac{17}{3} \frac{2}{3} = \frac{34}{9}$

22.  $\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n = 1 + \frac{2}{5} + \frac{4}{25} + \dots$

$S_0 = 1, S_1 = \frac{7}{5}, \dots$

Matches (e). The series is geometric:

$\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n = \frac{1}{1-(2/5)} = \frac{5}{3}$

23.  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots, S_n = 1 - \frac{1}{n+1}$

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = 1$

$$24. \sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \left( \frac{1}{2n} - \frac{1}{2(n+2)} \right) = \left( \frac{1}{2} - \frac{1}{6} \right) + \left( \frac{1}{4} - \frac{1}{8} \right) + \left( \frac{1}{6} - \frac{1}{10} \right) + \left( \frac{1}{8} - \frac{1}{12} \right) + \left( \frac{1}{10} - \frac{1}{14} \right) + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[ \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} \right] = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$25. \sum_{n=0}^{\infty} 2 \left( \frac{3}{4} \right)^n$$

Geometric series with  $r = \frac{3}{4} < 1$   
Converges by Theorem 9.6

$$26. \sum_{n=0}^{\infty} 2 \left( -\frac{1}{2} \right)^n$$

Geometric series with  $|r| = \left| -\frac{1}{2} \right| < 1$   
Converges by Theorem 9.6

$$27. \sum_{n=0}^{\infty} (0.9)^n$$

Geometric series with  $r = 0.9 < 1$   
Converges by Theorem 9.6

$$28. \sum_{n=0}^{\infty} (-0.6)^n$$

Geometric series with  $|r| = |-0.6| < 1$   
Converges by Theorem 9.6

$$29. (a) \sum_{n=1}^{\infty} \frac{6}{n(n+3)} = 2 \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+3} \right)$$

$$= 2 \left[ \left( 1 - \frac{1}{4} \right) + \left( \frac{1}{2} - \frac{1}{5} \right) + \left( \frac{1}{3} - \frac{1}{6} \right) + \left( \frac{1}{4} - \frac{1}{7} \right) + \cdots \right]$$

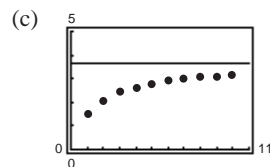
$$\left( S_n = 2 \left[ 1 + \frac{1}{2} + \frac{1}{3} - \left( \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} \right) \right] \right)$$

$$= 2 \left[ 1 + \frac{1}{2} + \frac{1}{3} \right] = \frac{11}{3} \approx 3.667$$

(b)

| $n$   | 5      | 10     | 20     | 50     | 100    |
|-------|--------|--------|--------|--------|--------|
| $S_n$ | 2.7976 | 3.1643 | 3.3936 | 3.5513 | 3.6078 |

(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.



$$30. (a) \sum_{n=1}^{\infty} \frac{4}{n(n+4)} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+4} \right)$$

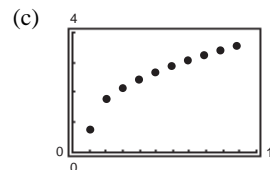
$$= \left( 1 - \frac{1}{5} \right) + \left( \frac{1}{2} - \frac{1}{6} \right) + \left( \frac{1}{3} - \frac{1}{7} \right) + \left( \frac{1}{4} - \frac{1}{8} \right) + \left( \frac{1}{5} - \frac{1}{9} \right) + \left( \frac{1}{6} - \frac{1}{10} \right) + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \approx 2.0833$$

(b)

| $n$   | 5      | 10     | 20     | 50     | 100    |
|-------|--------|--------|--------|--------|--------|
| $S_n$ | 1.5377 | 1.7607 | 1.9051 | 2.0071 | 2.0443 |

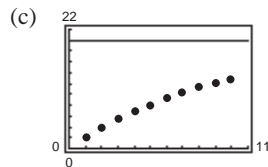
(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.



$$31. (a) \sum_{n=1}^{\infty} 2(0.9)^{n-1} = \sum_{n=0}^{\infty} 2(0.9)^n = \frac{2}{1-0.9} = 20$$

|     |       |        |         |         |         |         |
|-----|-------|--------|---------|---------|---------|---------|
| (b) | $n$   | 5      | 10      | 20      | 50      | 100     |
|     | $S_n$ | 8.1902 | 13.0264 | 17.5685 | 19.8969 | 19.9995 |

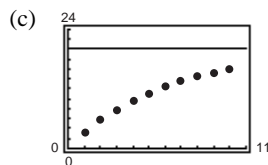
(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.



$$32. (a) \sum_{n=1}^{\infty} 3(0.85)^{n-1} = \frac{3}{1-0.85} = 20 \quad (\text{Geometric series})$$

|     |       |         |         |         |         |           |
|-----|-------|---------|---------|---------|---------|-----------|
| (b) | $n$   | 5       | 10      | 20      | 50      | 100       |
|     | $S_n$ | 11.1259 | 16.0625 | 19.2248 | 19.9941 | 19.999998 |

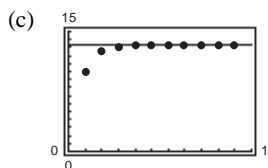
(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.



$$33. (a) \sum_{n=1}^{\infty} 10(0.25)^{n-1} = \frac{10}{1-0.25} = \frac{40}{3} \approx 13.3333$$

|     |       |         |         |         |         |         |
|-----|-------|---------|---------|---------|---------|---------|
| (b) | $n$   | 5       | 10      | 20      | 50      | 100     |
|     | $S_n$ | 13.3203 | 13.3333 | 13.3333 | 13.3333 | 13.3333 |

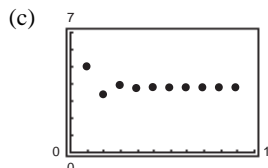
(d) The terms of the series decrease in magnitude rapidly. Thus, the sequence of partial sums approaches the sum rapidly.



$$34. (a) \sum_{n=1}^{\infty} 5\left(-\frac{1}{3}\right)^{n-1} = \frac{5}{1-(-1/3)} = \frac{15}{4} = 3.75$$

|     |       |        |        |        |        |        |
|-----|-------|--------|--------|--------|--------|--------|
| (b) | $n$   | 5      | 10     | 20     | 50     | 100    |
|     | $S_n$ | 3.7654 | 3.7499 | 3.7500 | 3.7500 | 3.7500 |

(d) The terms of the series decrease in magnitude rapidly. Thus, the sequence of partial sums approaches the sum rapidly.



$$35. \sum_{n=2}^{\infty} \frac{1}{n^2-1} = \sum_{n=2}^{\infty} \left( \frac{1/2}{n-1} - \frac{1/2}{n+1} \right) = \frac{1}{2} \sum_{n=2}^{\infty} \left( \frac{1}{n-1} - \frac{1}{n+1} \right)$$

$$= \frac{1}{2} \left[ \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots \right]$$

$$= \frac{1}{2} \left(1 + \frac{1}{2}\right) = \frac{3}{4}$$

$$36. \sum_{n=1}^{\infty} \frac{4}{n(n+2)} = 2 \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+2} \right) = 2 \left[ \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots \right] = 2 \left(1 + \frac{1}{2}\right) = 3$$

$$37. \sum_{n=1}^{\infty} \frac{8}{(n+1)(n+2)} = 8 \sum_{n=1}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+2} \right) = 8 \left[ \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots \right] = 8 \left(\frac{1}{2}\right) = 4$$

$$38. \sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)} = \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{1}{2n+1} - \frac{1}{2n+3} \right) = \frac{1}{2} \left[ \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{9}\right) + \cdots \right] = \frac{1}{2} \left(\frac{1}{3}\right) = \frac{1}{6}$$

$$39. \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - (1/2)} = 2$$

$$40. \sum_{n=0}^{\infty} 6\left(\frac{4}{5}\right)^n = \frac{6}{1 - (4/5)} = 30$$

$$41. \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{1}{1 - (-1/2)} = \frac{2}{3}$$

(Geometric)

$$42. \sum_{n=0}^{\infty} 2\left(-\frac{2}{3}\right)^n = \frac{2}{1 - (-2/3)} = \frac{6}{5}$$

$$43. \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n = \frac{1}{1 - (1/10)} = \frac{10}{9}$$

$$44. \sum_{n=0}^{\infty} 8\left(\frac{3}{4}\right)^n = \frac{8}{1 - (3/4)} = 32$$

$$45. \sum_{n=0}^{\infty} 3\left(-\frac{1}{3}\right)^n = \frac{3}{1 - (-1/3)} = \frac{9}{4}$$

$$46. \sum_{n=0}^{\infty} 4\left(-\frac{1}{2}\right)^n = \frac{4}{1 - (-1/2)} = \frac{8}{3}$$

$$\begin{aligned} 47. \sum_{n=0}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n}\right) &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n \\ &= \frac{1}{1 - (1/2)} - \frac{1}{1 - (1/3)} \\ &= 2 - \frac{3}{2} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} 48. \sum_{n=1}^{\infty} [(0.7)^n + (0.9)^n] &= \sum_{n=0}^{\infty} \left(\frac{7}{10}\right)^n + \sum_{n=0}^{\infty} \left(\frac{9}{10}\right)^n - 2 \\ &= \frac{1}{1 - (7/10)} + \frac{1}{1 - (9/10)} - 2 \\ &= \frac{10}{3} + 10 - 2 = \frac{34}{3} \end{aligned}$$

49. Note that  $\sin(1) \approx 0.8415 < 1$ . The series  $\sum_{n=1}^{\infty} [\sin(1)]^n$  is geometric with  $r = \sin(1) < 1$ . Thus,

$$\sum_{n=1}^{\infty} [\sin(1)]^n = \sin(1) \sum_{n=0}^{\infty} [\sin(1)]^n = \frac{\sin(1)}{1 - \sin(1)} \approx 5.3080.$$

$$\begin{aligned} 50. S_n &= \sum_{k=1}^n \frac{1}{9k^2 + 3k - 2} = \sum_{k=1}^n \frac{1}{(3k-1)(3k+2)} \\ &= \sum_{k=1}^n \left[ \frac{1}{9k-3} - \frac{1}{9k+6} \right] = \frac{1}{3} \sum_{k=1}^n \left[ \frac{1}{3k-1} - \frac{1}{3k+2} \right] \\ &= \frac{1}{3} \left[ \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{8}\right) + \left(\frac{1}{8} - \frac{1}{11}\right) + \cdots + \left(\frac{1}{3n-1} - \frac{1}{3n+2}\right) \right] \\ &= \frac{1}{3} \left[ \frac{1}{2} - \frac{1}{3n+2} \right] \end{aligned}$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{3} \left[ \frac{1}{2} - \frac{1}{3n+2} \right] = \frac{1}{6}$$

$$51. (a) 0.\bar{4} = \sum_{n=0}^{\infty} \frac{4}{10} \left(\frac{1}{10}\right)^n$$

(b) Geometric series with  $a = \frac{4}{10}$  and  $r = \frac{1}{10}$

$$S = \frac{a}{1-r} = \frac{4/10}{1 - (1/10)} = \frac{4}{9}$$

$$52. (a) 0.\bar{9} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

$$= \sum_{n=1}^{\infty} 9\left(\frac{1}{10}\right)^n = \frac{9}{10} \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n$$

$$(b) 0.\bar{9} = \frac{9}{10} \frac{1}{1 - (1/10)} = 1$$

$$53. (a) 0.\overline{81} = \sum_{n=0}^{\infty} \frac{81}{100} \left(\frac{1}{100}\right)^n$$

(b) Geometric series with  $a = \frac{81}{100}$  and  $r = \frac{1}{100}$

$$S = \frac{a}{1-r} = \frac{81/100}{1 - (1/100)} = \frac{81}{99} = \frac{9}{11}$$

$$54. (a) 0.\overline{01} = \sum_{n=1}^{\infty} \left(\frac{1}{100}\right)^n = \frac{1}{100} \sum_{n=0}^{\infty} \left(\frac{1}{100}\right)^n$$

$$(b) 0.\overline{01} = \frac{1}{100} \cdot \frac{1}{1 - (1/100)} = \frac{1}{100} \cdot \frac{100}{99} = \frac{1}{99}$$

$$55. \text{ (a) } 0.0\overline{75} = \sum_{n=0}^{\infty} \frac{3}{40} \left( \frac{1}{100} \right)^n$$

(b) Geometric series with  $a = \frac{3}{40}$  and  $r = \frac{1}{100}$

$$S = \frac{a}{1-r} = \frac{3/40}{99/100} = \frac{5}{66}$$

$$57. \sum_{n=1}^{\infty} \frac{n+10}{10n+1}$$

$$\lim_{n \rightarrow \infty} \frac{n+10}{10n+1} = \frac{1}{10} \neq 0$$

Diverges by Theorem 9.9

$$59. \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+2} \right) = \left( 1 - \frac{1}{3} \right) + \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \left( \frac{1}{4} - \frac{1}{6} \right) + \cdots = 1 + \frac{1}{2} = \frac{3}{2}, \text{ converges}$$

$$60. \sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \frac{1}{3} \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+3} \right)$$

$$= \frac{1}{3} \left[ \left( 1 - \frac{1}{4} \right) + \left( \frac{1}{2} - \frac{1}{5} \right) + \left( \frac{1}{3} - \frac{1}{6} \right) + \left( \frac{1}{4} - \frac{1}{7} \right) + \left( \frac{1}{5} - \frac{1}{8} \right) + \left( \frac{1}{6} - \frac{1}{9} \right) + \cdots \right]$$

$$= \frac{1}{3} \left( 1 + \frac{1}{2} + \frac{1}{3} \right) = \frac{11}{18}, \text{ converges}$$

$$61. \sum_{n=1}^{\infty} \frac{3n-1}{2n+1}$$

$$\lim_{n \rightarrow \infty} \frac{3n-1}{2n+1} = \frac{3}{2} \neq 0$$

Diverges by Theorem 9.9

$$62. \sum_{n=1}^{\infty} \frac{3^n}{n^3}$$

$$\lim_{n \rightarrow \infty} \frac{3^n}{n^3} = \lim_{n \rightarrow \infty} \frac{(\ln 2)3^n}{3n^2}$$

$$= \lim_{n \rightarrow \infty} \frac{(\ln 2)^2 3^n}{6n} = \lim_{n \rightarrow \infty} \frac{(\ln n)^3 3^n}{6} = \infty$$

(by L'Hôpital's Rule); diverges by Theorem 9.9

$$63. \sum_{n=0}^{\infty} \frac{4}{2^n} = 4 \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n$$

Geometric series with  $r = \frac{1}{2}$

Converges by Theorem 9.6

$$64. \sum_{n=1}^{\infty} \frac{1}{4^n}$$

Geometric series with  $r = \frac{1}{4}$

Converges by Theorem 9.6

$$65. \sum_{n=0}^{\infty} (1.075)^n$$

Geometric series with  $r = 1.075$

Diverges by Theorem 9.6

$$66. \sum_{n=1}^{\infty} \frac{2^n}{100}$$

Geometric series with  $r = 2$

Diverges by Theorem 9.6

$$67. \text{ Since } n > \ln(n), \text{ the terms } a_n = \frac{n}{\ln(n)}$$

do not approach 0 as  $n \rightarrow \infty$ . Hence, the series

$$\sum_{n=2}^{\infty} \frac{n}{\ln(n)} \text{ diverges.}$$

$$68. S_n = \sum_{k=1}^n \ln\left(\frac{1}{k}\right) = \sum_{k=1}^n -\ln(k)$$

$$= 0 - \ln 2 - \ln 3 - \cdots - \ln(n)$$

Since  $\lim_{n \rightarrow \infty} S_n$  diverges,  $\sum_{n=1}^{\infty} \ln\left(\frac{1}{n}\right)$  diverges.

69. For  $k \neq 0$ ,

$$\begin{aligned}\lim_{n \rightarrow \infty} \left(1 + \frac{k}{n}\right)^n &= \lim_{n \rightarrow \infty} \left[\left(1 + \frac{k}{n}\right)^{n/k}\right]^k \\ &= e^k \neq 0.\end{aligned}$$

For  $k = 0$ ,  $\lim_{n \rightarrow \infty} (1 + 0)^n = 1 \neq 0$ .

Hence,  $\sum_{n=1}^{\infty} \left[1 + \frac{k}{n}\right]^n$  diverges.

72.  $S_n = \sum_{k=1}^n \ln\left(\frac{k+1}{k}\right)$

$$= \ln\left(\frac{2}{1}\right) + \ln\left(\frac{3}{2}\right) + \cdots + \ln\left(\frac{n+1}{n}\right)$$

$$= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln(n+1) - \ln n)$$

$$= \ln(n+1) - \ln(1) = \ln(n+1)$$

Diverges

73. See definition on page 606.

74.  $\lim_{n \rightarrow \infty} a_n = 5$  means that the limit of the sequence  $\{a_n\}$  is 5.

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots = 5$  means that the limit of the partial sums is 5.

76. If  $\lim_{n \rightarrow \infty} a_n \neq 0$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.

77.  $a_n = \frac{n+1}{n}$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n+1}{n} = 1 \quad (\{a_n\} \text{ converges})$$

$\sum_{n=1}^{\infty} a_n$  diverges because  $\lim_{n \rightarrow \infty} a_n \neq 0$ .

79.  $\sum_{n=1}^{\infty} \frac{x^n}{2^n} = \sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n = \frac{x}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$

Geometric series: converges for  $\left|\frac{x}{2}\right| < 1$  or  $|x| < 2$

$$f(x) = \frac{x}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

$$= \frac{x}{2} \frac{1}{1 - (x/2)} = \frac{x}{2} \frac{2}{2 - x} = \frac{x}{2 - x}, \quad |x| < 2$$

70.  $\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$  converges since it is geometric with

$$|r| = \frac{1}{e} < 1.$$

71.  $\lim_{n \rightarrow \infty} \arctan n = \frac{\pi}{2} \neq 0$

Hence,  $\sum_{n=1}^{\infty} \arctan n$  diverges.

75. The series given by

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots, \quad a \neq 0$$

is a geometric series with ratio  $r$ . When  $0 < |r| < 1$ , the series converges to  $a/(1 - r)$ . The series diverges if  $|r| \geq 1$ .

78. (a)  $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$

(b)  $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$

These are the same. The third series is different, unless  $a_1 = a_2 = \cdots = a$  is constant.

(c)  $\sum_{n=1}^{\infty} a_k = a_k + a_k + \cdots$

80.  $\sum_{n=1}^{\infty} (3x)^n = (3x) \sum_{n=0}^{\infty} (3x)^n$

Geometric series: converges for  $|3x| < 1 \Rightarrow |x| < \frac{1}{3}$

$$f(x) = (3x) \sum_{n=0}^{\infty} (3x)^n = (3x) \frac{1}{1 - 3x} = \frac{3x}{1 - 3x}, \quad |x| < \frac{1}{3}$$

$$81. \sum_{n=1}^{\infty} (x-1)^n = (x-1) \sum_{n=0}^{\infty} (x-1)^n$$

Geometric series: converges for  $|x-1| < 1 \Rightarrow 0 < x < 2$

$$f(x) = (x-1) \sum_{n=0}^{\infty} (x-1)^n$$

$$= (x-1) \frac{1}{1-(x-1)} = \frac{x-1}{2-x}, \quad 0 < x < 2$$

$$83. \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n$$

Geometric series: converges for

$$|-x| < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$$

$$f(x) = \sum_{n=0}^{\infty} (-x)^n = \frac{1}{1+x}, \quad -1 < x < 1$$

$$85. \sum_{n=0}^{\infty} \left(\frac{1}{x}\right)^n$$

Geometric series: converges if  $\left|\frac{1}{x}\right| < 1$

$$\Rightarrow |x| > 1 \Rightarrow x < -1 \text{ or } x > 1$$

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{1}{x}\right)^n = \frac{1}{1-(1/x)} = \frac{x}{x-1}, \quad x > 1 \text{ or } x < -1$$

87. (a) Yes, the new series will still diverge.

(b) Yes, the new series will converge.

89. (a)  $x$  is the common ratio.

$$(b) 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1$$

90. (a)  $\left(-\frac{x}{2}\right)$  is the common ratio.

$$(b) 1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots = \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n$$

$$= \frac{1}{1 - (-x/2)}$$

$$= \frac{2}{2+x}, \quad |x| < 2$$

$$82. \sum_{n=0}^{\infty} 4 \left(\frac{x-3}{4}\right)^n$$

Geometric series: converges for

$$\left|\frac{x-3}{4}\right| < 1 \Rightarrow |x-3| < 4 \Rightarrow -1 < x < 7$$

$$f(x) = \sum_{n=0}^{\infty} 4 \left(\frac{x-3}{4}\right)^n = \frac{4}{1 - [(x-3)/4]}$$

$$= \frac{4}{(4-x+3)/4} = \frac{16}{7-x}, \quad -1 < x < 7$$

$$84. \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n$$

Geometric series: converges for

$$|-x^2| < 1 \Rightarrow -1 < x < 1$$

$$f(x) = \sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1 - (-x^2)} = \frac{1}{1+x^2}, \quad -1 < x < 1$$

$$86. \sum_{n=1}^{\infty} \left(\frac{x^2}{x^2+4}\right)^n = \frac{x^2}{x^2+4} \sum_{n=0}^{\infty} \left(\frac{x^2}{x^2+4}\right)^n$$

Geometric series: converges for  $\left|\frac{x^2}{x^2+4}\right| < 1$

$$\Rightarrow x^2 < x^2 + 4 \Rightarrow \text{converges for all } x$$

$$f(x) = \frac{x^2}{x^2+4} \cdot \frac{1}{1 - \frac{x^2}{x^2+4}} = \frac{x^2}{x^2+4} \cdot \frac{x^2+4}{4} = \frac{x^2}{4}$$

88. Neither statement is true. The formula

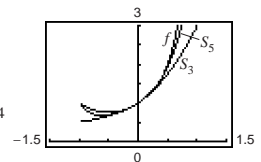
$$\frac{1}{x-1} = 1 + x + x^2 + \cdots$$

holds for  $-1 < x < 1$ .

$$(c) y_1 = \frac{1}{1-x}$$

$$y_2 = S_3 = 1 + x + x^2$$

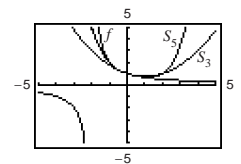
$$y_3 = S_5 = 1 + x + x^2 + x^3 + x^4$$



$$(c) y_1 = \frac{2}{2+x}$$

$$y_2 = S_3 = 1 - \frac{x}{2} + \frac{x^2}{4}$$

$$y_3 = S_5 = 1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{16}$$



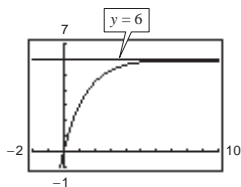
$$91. f(x) = 3 \left[ \frac{1 - 0.5^x}{1 - 0.5} \right]$$

Horizontal asymptote:  $y = 6$

$$\sum_{n=0}^{\infty} 3 \left( \frac{1}{2} \right)^n$$

$$S = \frac{3}{1 - (1/2)} = 6$$

The horizontal asymptote is the sum of the series.  $f(n)$  is the  $n$ th partial sum.



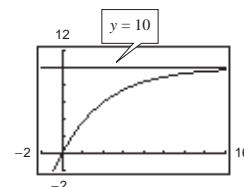
$$92. f(x) = 2 \left[ \frac{1 - 0.8^x}{1 - 0.8} \right]$$

Horizontal asymptote:  $y = 10$

$$\sum_{n=0}^{\infty} 2 \left( \frac{4}{5} \right)^n$$

$$S = \frac{2}{1 - (4/5)} = 10$$

The horizontal asymptote is the sum of the series.  $f(n)$  is the  $n$ th partial sum.



$$93. \frac{1}{n(n+1)} < 0.0001$$

$$10,000 < n^2 + n$$

$$0 < n^2 + n - 10,000$$

$$n = \frac{-1 \pm \sqrt{1^2 - 4(1)(-10,000)}}{2}$$

Choosing the positive value for  $n$  we have  $n \approx 99.5012$ . The first term that is less than 0.0001 is  $n = 100$ .

$$\left( \frac{1}{8} \right)^n < 0.0001$$

$$10,000 < 8^n$$

This inequality is true when  $n = 5$ . This series converges at a faster rate.

$$94. \frac{1}{2^n} < 0.0001$$

$$10,000 < 2^n$$

This inequality is true when  $n = 14$ .

$$(0.01)^n < 0.0001$$

$$10,000 < 10^n$$

This inequality is true when  $n = 5$ . This series converges at a faster rate.

$$95. \sum_{i=0}^{n-1} 8000(0.9)^i = \frac{8000[1 - (0.9)^{(n-1)+1}]}{1 - 0.9}$$

$$= 80,000(1 - 0.9^n), \quad n > 0$$

$$97. \sum_{i=0}^{n-1} 100(0.75)^i = \frac{100[1 - 0.75^{(n-1)+1}]}{1 - 0.75}$$

$$= 400(1 - 0.75^n) \text{ million dollars}$$

Sum = 400 million dollars

$$99. D_1 = 16$$

$$D_2 = \underbrace{0.81(16)}_{\text{up}} + \underbrace{0.81(16)}_{\text{down}} = 32(0.81)$$

$$D_3 = 16(0.81)^2 + 16(0.81)^2 = 32(0.81)^2$$

$\vdots$

$$D = 16 + 32(0.81) + 32(0.81)^2 + \cdots$$

$$= -16 + \sum_{n=0}^{\infty} 32(0.81)^n = -16 + \frac{32}{1 - 0.81}$$

$$\approx 152.42 \text{ feet}$$

$$96. V(t) = 225,000(1 - 0.3)^n = (0.7)^n(225,000)$$

$$V(5) = (0.7)^5(225,000) = \$37,815.75$$

$$98. \sum_{i=0}^{n-1} 100(0.60)^i = \frac{100[1 - 0.6^n]}{1 - 0.6}$$

$$= 250(1 - 0.6^n) \text{ million dollars}$$

Sum = 250 million dollars

100. The ball in Exercise 99 takes the following times for each fall.

$$s_1 = -16t^2 + 16$$

$$s_1 = 0 \text{ if } t = 1$$

$$s_2 = -16t^2 + 16(0.81)$$

$$s_2 = 0 \text{ if } t = 0.9$$

$$s_3 = -16t^2 + 16(0.81)^2$$

$$s_3 = 0 \text{ if } t = (0.9)^2$$

$\vdots$

$\vdots$

$$s_n = -16t^2 + 16(0.81)^{n-1}$$

$$s_n = 0 \text{ if } t = (0.9)^{n-1}$$

Beginning with  $s_2$ , the ball takes the same amount of time to bounce up as it takes to fall. The total elapsed time before the ball comes to rest is

$$t = 1 + 2 \sum_{n=1}^{\infty} (0.9)^n = -1 + 2 \sum_{n=0}^{\infty} (0.9)^n$$

$$= -1 + \frac{2}{1 - 0.9} = 19 \text{ seconds.}$$



$$101. P(n) = \frac{1}{2} \left( \frac{1}{2} \right)^n$$

$$P(2) = \frac{1}{2} \left( \frac{1}{2} \right)^2 = \frac{1}{8}$$

$$\sum_{n=0}^{\infty} \frac{1}{2} \left( \frac{1}{2} \right)^n = \frac{1/2}{1 - (1/2)} = 1$$

$$102. P(n) = \frac{1}{3} \left( \frac{2}{3} \right)^n$$

$$P(2) = \frac{1}{3} \left( \frac{2}{3} \right)^2 = \frac{4}{27}$$

$$\sum_{n=0}^{\infty} \frac{1}{3} \left( \frac{2}{3} \right)^n = \frac{1/3}{1 - (2/3)} = 1$$

$$103. (a) \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n = \sum_{n=0}^{\infty} \frac{1}{2} \left( \frac{1}{2} \right)^n = \frac{1}{2} \frac{1}{1 - (1/2)} = 1$$

(b) No, the series is not geometric.

$$(c) \sum_{n=1}^{\infty} n \left( \frac{1}{2} \right)^n = 2$$

$$104. \text{ Person 1: } \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^7} + \cdots = \frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{1}{8} \right)^n = \frac{1}{2} \frac{1}{1 - (1/8)} = \frac{4}{7}$$

$$\text{Person 2: } \frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^8} + \cdots = \frac{1}{4} \sum_{n=0}^{\infty} \left( \frac{1}{8} \right)^n = \frac{1}{4} \frac{1}{1 - (1/8)} = \frac{2}{7}$$

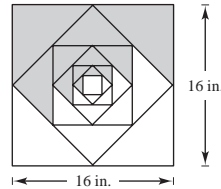
$$\text{Person 3: } \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \cdots = \frac{1}{8} \sum_{n=0}^{\infty} \left( \frac{1}{8} \right)^n = \frac{1}{8} \frac{1}{1 - (1/8)} = \frac{1}{7}$$

$$\text{Sum: } \frac{4}{7} + \frac{2}{7} + \frac{1}{7} = 1$$

$$105. (a) 64 + 32 + 16 + 8 + 4 + 2 = 126 \text{ in.}^2$$

$$(b) \sum_{n=0}^{\infty} 64 \left( \frac{1}{2} \right)^n = \frac{64}{1 - (1/2)} = 128 \text{ in.}^2$$

**Note:** This is one-half of the area of the original square!



$$106. (a) \sin \theta = \frac{|Y_{y_1}|}{z} \Rightarrow |Y_{y_1}| = z \sin \theta$$

$$\sin \theta = \frac{|x_1 y_1|}{|Y_{y_1}|} \Rightarrow |x_1 y_1| = |Y_{y_1}| \sin \theta = z \sin^2 \theta$$

$$\sin \theta = \frac{|x_1 y_2|}{|x_1 y_1|} \Rightarrow |x_1 y_2| = |x_1 y_1| \sin \theta = z \sin^3 \theta$$

$$\text{Total: } z \sin \theta + z \sin^2 \theta + z \sin^3 \theta + \cdots = z \frac{\sin \theta}{1 - \sin \theta}$$

$$(b) \text{ If } z = 1 \text{ and } \theta = \frac{\pi}{6}, \text{ then total} = \frac{1/2}{1 - (1/2)} = 1.$$

$$107. \sum_{n=1}^{20} 50,000 \left( \frac{1}{1.06} \right)^n = \frac{50,000}{1.06} \sum_{i=0}^{19} \left( \frac{1}{1.06} \right)^i = \frac{50,000}{1.06} \left[ \frac{1 - 1.06^{-20}}{1 - 1.06^{-1}} \right], \quad [n = 20, r = 1.06^{-1}]$$

$$\approx 573,496.06$$

$$108. \text{ Surface area} = 4\pi(1)^2 + 9\left(4\pi\left(\frac{1}{3}\right)^2\right) + 9^2 \cdot 4\pi\left(\frac{1}{9}\right)^2 + \cdots = 4[\pi + \pi + \cdots] = \infty$$

$$109. w = \sum_{i=0}^{n-1} 0.01(2)^i = \frac{0.01(1 - 2^n)}{1 - 2} = 0.01(2^n - 1)$$

(a) When  $n = 29$ :  $w = \$5,368,709.11$

(b) When  $n = 30$ :  $w = \$10,737,418.23$

(c) When  $n = 31$ :  $w = \$21,474,836.47$

$$110. \sum_{n=0}^{12t-1} P \left(1 + \frac{r}{12}\right)^n = \frac{P \left[1 - \left(1 + \frac{r}{12}\right)^{12t}\right]}{1 - \left(1 + \frac{r}{12}\right)} = P \left(-\frac{12}{r}\right) \left[1 - \left(1 + \frac{r}{12}\right)^{12t}\right] = P \left(\frac{12}{r}\right) \left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]$$

$$\sum_{n=0}^{12t-1} P(e^{r/12})^n = \frac{P(1 - (e^{r/12})^{12t})}{1 - e^{r/12}} = \frac{P(e^{rt} - 1)}{e^{r/12} - 1}$$

111.  $P = 50, r = 0.03, t = 20$

(a)  $A = 50 \left(\frac{12}{0.03}\right) \left[\left(1 + \frac{0.03}{12}\right)^{12(20)} - 1\right] \approx \$16,415.10$

(b)  $A = \frac{50(e^{0.03(20)} - 1)}{e^{0.03/12} - 1} \approx \$16,421.83$

113.  $P = 100, r = 0.04, t = 40$

(a)  $A = 100 \left(\frac{12}{0.04}\right) \left[\left(1 + \frac{0.04}{12}\right)^{12(40)} - 1\right] \approx \$118,196.13$

(b)  $A = \frac{100(e^{0.04(40)} - 1)}{e^{0.04/12} - 1} \approx \$118,393.43$

112.  $P = 75, r = 0.05, t = 25$

(a)  $A = 75 \left(\frac{12}{0.05}\right) \left[\left(1 + \frac{0.05}{12}\right)^{12(25)} - 1\right] \approx \$44,663.23$

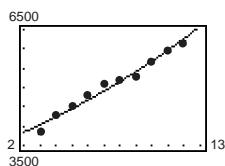
(b)  $A = \frac{75(e^{0.05(25)} - 1)}{e^{0.05/12} - 1} \approx \$44,732.85$

114.  $P = 20, r = 0.06, t = 50$

(a)  $A = 20 \left(\frac{12}{0.06}\right) \left[\left(1 + \frac{0.06}{12}\right)^{12(50)} - 1\right] \approx \$75,743.82$

(b)  $A = \frac{20(e^{0.06(50)} - 1)}{e^{0.06/12} - 1} \approx \$76,151.45$

115. (a)  $a_n = 3484.1363(1.0502)^n = 3484.1363e^{0.04897n}$



(b) Adding the ten values  $a_3, a_4, \dots, a_{12}$  you obtain

$$\sum_{n=3}^{12} a_n = 50,809, \text{ or } \$50,809,000,000.$$

(c)  $\sum_{n=3}^{12} 3484.1363e^{0.04897n} \approx 50,803, \text{ or } \$50,803,000,000$

(Answers will vary.)

116.  $T = 40,000 + 40,000(1.04) + \dots + 40,000(1.04)^{39}$

$$= \sum_{n=0}^{39} 40,000(1.04)^n = 40,000 \left(\frac{1 - 1.04^{40}}{1 - 1.04}\right) \approx \$3,801,020$$

117. False.  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ , but  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges.

118. True

119. False;  $\sum_{n=1}^{\infty} ar^n = \left(\frac{a}{1-r}\right) - a$

120. True

The formula requires that the geometric series begins with  $n = 0$ .

$$\lim_{n \rightarrow \infty} \frac{n}{1000(n+1)} = \frac{1}{1000} \neq 0$$

121. True

$$\begin{aligned} 0.74999 \dots &= 0.74 + \frac{9}{10^3} + \frac{9}{10^4} + \dots \\ &= 0.74 + \frac{9}{10^3} \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n \\ &= 0.74 + \frac{9}{10^3} \cdot \frac{1}{1 - (1/10)} \\ &= 0.74 + \frac{9}{10^3} \cdot \frac{10}{9} \\ &= 0.74 + \frac{1}{100} = 0.75 \end{aligned}$$

122. True

123. By letting  $S_0 = 0$ , we have

$$\begin{aligned} a_n &= \sum_{k=1}^n a_k - \sum_{k=1}^{n-1} a_k = S_n - S_{n-1}. \text{ Thus,} \\ \sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} (S_n - S_{n-1}) \\ &= \sum_{n=1}^{\infty} (S_n - S_{n-1} + c - c) \\ &= \sum_{n=1}^{\infty} [(c - S_{n-1}) - (c - S_n)]. \end{aligned}$$

125. Let  $\sum a_n = \sum_{n=0}^{\infty} 1$  and  $\sum b_n = \sum_{n=0}^{\infty} (-1)$ .

Both are divergent series.

$$\sum (a_n + b_n) = \sum_{n=0}^{\infty} [1 + (-1)] = \sum_{n=0}^{\infty} [1 - 1] = 0$$

127. Suppose, on the contrary, that  $\sum ca_n$  converges. Because  $c \neq 0$ ,

$$\sum \left(\frac{1}{c}\right) ca_n = \sum a_n$$

converges. This is a contradiction since  $\sum a_n$  diverged. Hence,  $\sum ca_n$  diverges.

$$\begin{aligned} 129. \text{ (a)} \quad \frac{1}{a_{n+1}a_{n+2}} - \frac{1}{a_{n+2}a_{n+3}} &= \frac{a_{n+3} - a_{n+1}}{a_{n+1}a_{n+2}a_{n+3}} \\ &= \frac{a_{n+2}}{a_{n+1}a_{n+2}a_{n+3}} \\ &= \frac{1}{a_{n+1}a_{n+3}} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad S_n &= \sum_{k=0}^n \frac{1}{a_{k+1}a_{k+3}} \\ &= \sum_{k=0}^n \left[ \frac{1}{a_{k+1}a_{k+2}} - \frac{1}{a_{k+2}a_{k+3}} \right] \\ &= \left[ \frac{1}{a_1a_2} - \frac{1}{a_2a_3} \right] + \left[ \frac{1}{a_2a_3} - \frac{1}{a_3a_4} \right] + \cdots + \left[ \frac{1}{a_{n+1}a_{n+2}} - \frac{1}{a_{n+2}a_{n+3}} \right] \\ &= \frac{1}{a_1a_2} - \frac{1}{a_{n+2}a_{n+3}} \\ &= 1 - \frac{1}{a_{n+2}a_{n+3}} \\ \sum_{n=0}^{\infty} \frac{1}{a_{n+1}a_{n+3}} &= \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[ 1 - \frac{1}{a_{n+2}a_{n+3}} \right] = 1 \end{aligned}$$

124. Let  $\{S_n\}$  be the sequence of partial sums for the convergent series

$$\sum_{n=1}^{\infty} a_n = L. \text{ Then } \lim_{n \rightarrow \infty} S_n = L \text{ and since}$$

$$R_n = \sum_{k=n+1}^{\infty} a_k = L - S_n,$$

we have

$$\begin{aligned} \lim_{n \rightarrow \infty} R_n &= \lim_{n \rightarrow \infty} (L - S_n) \\ &= \lim_{n \rightarrow \infty} L - \lim_{n \rightarrow \infty} S_n = L - L = 0. \end{aligned}$$

126. If  $\sum (a_n + b_n)$  converged, then  $\sum (a_n + b_n) - \sum a_n = \sum b_n$  would converge, which is a contradiction. Thus,  $\sum (a_n + b_n)$  diverges.

128. If  $\sum a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ . Hence,

$$\lim_{n \rightarrow \infty} \frac{1}{a_n} \neq 0,$$

which implies that  $\sum 1/a_n$  diverges.

$$\begin{aligned}
 130. S_{2n} &= 1 + 2x + x^2 + 2x^3 + \cdots + x^{2n} + 2x^{2n+1} \\
 &= (1 + x^2 + x^4 + \cdots + x^{2n}) + (2x + 2x^3 + \cdots + 2x^{2n+1}) \\
 &= \sum_{k=0}^n (x^2)^k + 2x \sum_{k=0}^n (x^2)^k
 \end{aligned}$$

As  $n \rightarrow \infty$ , the two geometric series converge for  $|x| < 1$ :

$$\lim_{n \rightarrow \infty} S_{2n} = \frac{1}{1-x^2} + 2x \left( \frac{1}{1-x^2} \right) = \frac{1+2x}{1-x^2}, \quad |x| < 1$$

$$\begin{aligned}
 131. \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \cdots &= \sum_{n=0}^{\infty} \frac{1}{r} \left( \frac{1}{r} \right)^n \\
 &= \frac{1/r}{1 - (1/r)} \\
 &= \frac{1}{r-1} \quad \left( \text{since } \left| \frac{1}{r} \right| < 1 \right)
 \end{aligned}$$

This is a geometric series which converges if

$$\left| \frac{1}{r} \right| < 1 \iff |r| > 1.$$

133. Let  $H$  represent the half-life of the drug. If a patient receives  $n$  equal doses of  $P$  units each of this drug, administered at equal time interval of length  $t$ , the total amount of the drug in the patient's system at the time the last dose is administered is given by  $T_n = P + Pe^{kt} + Pe^{2kt} + \cdots + Pe^{(n-1)kt}$  where  $k = -(\ln 2)/H$ . One time interval *after* the last dose is administered is given by  $T_{n+1} = Pe^{kt} + Pe^{2kt} + Pe^{3kt} + \cdots + Pe^{nkt}$ . Two time intervals *after* the last dose is administered is given by  $T_{n+2} = Pe^{2kt} + Pe^{3kt} + Pe^{4kt} + \cdots + Pe^{(n+1)kt}$  and so on. Since  $k < 0$ ,  $T_{n+s} \rightarrow 0$  as  $s \rightarrow \infty$ , where  $s$  is an integer.

134. The series is telescoping:

$$\begin{aligned}
 S_n &= \sum_{k=1}^n \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)} \\
 &= \sum_{k=1}^n \left[ \frac{3^k}{3^k - 2^k} - \frac{3^{k+1}}{3^{k+1} - 2^{k+1}} \right] \\
 &= 3 - \frac{3^{n+1}}{3^{n+1} - 2^{n+1}} \\
 \lim_{n \rightarrow \infty} S_n &= 3 - 1 = 2
 \end{aligned}$$

132. The entire rectangle has area 2 because the height is 1 and the base is  $1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$ . The squares all lie inside the rectangle, and the sum of their areas is

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$$\text{Thus, } \sum_{n=1}^{\infty} \frac{1}{n^2} < 2.$$

135.  $f(1) = 0, f(2) = 1, f(3) = 2, f(4) = 4, \dots$

$$\text{In general: } f(n) = \begin{cases} n^2/4, & n \text{ even} \\ (n^2 - 1)/4, & n \text{ odd.} \end{cases}$$

(See below for a proof of this.)

$x + y$  and  $x - y$  are either both odd or both even. If both even, then

$$f(x+y) - f(x-y) = \frac{(x+y)^2}{4} - \frac{(x-y)^2}{4} = xy.$$

If both odd,

$$\begin{aligned}
 f(x+y) - f(x-y) &= \frac{(x+y)^2 - 1}{4} - \frac{(x-y)^2 - 1}{4} \\
 &= xy.
 \end{aligned}$$

Proof by induction that the formula for  $f(n)$  is correct.

It is true for  $n = 1$ . Assume that the formula is valid for  $k$ . If  $k$  is even, then  $f(k) = k^2/4$  and

$$\begin{aligned}
 f(k+1) &= f(k) + \frac{k}{2} = \frac{k^2}{4} + \frac{k}{2} \\
 &= \frac{k^2 + 2k}{4} = \frac{(k+1)^2 - 1}{4}.
 \end{aligned}$$

The argument is similar if  $k$  is odd.

Section 9.3 The Integral Test and  $p$ -Series

1. 
$$\sum_{n=1}^{\infty} \frac{1}{n+1}$$

Let  $f(x) = \frac{1}{x+1}$ .

 $f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{1}{x+1} dx = \left[ \ln(x+1) \right]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

2. 
$$\sum_{n=1}^{\infty} \frac{2}{3n+5}$$

Let  $f(x) = \frac{2}{3x+5}$ .

 $f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{2}{3x+5} dx = \left[ \frac{2}{3} \ln(3x+5) \right]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

3. 
$$\sum_{n=1}^{\infty} e^{-n}$$

Let  $f(x) = e^{-x}$ .

 $f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} e^{-x} dx = \left[ -e^{-x} \right]_1^{\infty} = \frac{1}{e}$$

Converges by Theorem 9.10

4. 
$$\sum_{n=1}^{\infty} ne^{-n/2}$$

Let  $f(x) = xe^{-x/2}$ .

 $f$  is positive, continuous, and decreasing for  $x \geq 3$  since

$$f'(x) = \frac{2-x}{2e^{x/2}} < 0 \text{ for } x \geq 3.$$

$$\int_3^{\infty} xe^{-x/2} dx = \left[ -2(x+2)e^{-x/2} \right]_3^{\infty} = 10e^{-3/2}$$

Converges by Theorem 9.10

5. 
$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

Let  $f(x) = \frac{1}{x^2+1}$ .

 $f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{1}{x^2+1} dx = \left[ \arctan x \right]_1^{\infty} = \frac{\pi}{4}$$

Converges by Theorem 9.10

6. 
$$\sum_{n=1}^{\infty} \frac{1}{2n+1}$$

Let  $f(x) = \frac{1}{2x+1}$ .

 $f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{1}{2x+1} dx = \left[ \ln \sqrt{2x+1} \right]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

7. 
$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$$

Let  $f(x) = \frac{\ln(x+1)}{x+1}$ .

 $f$  is positive, continuous, and decreasing for  $x \geq 2$  since

$$f'(x) = \frac{1 - \ln(x+1)}{(x+1)^2} < 0 \text{ for } x \geq 2.$$

$$\int_1^{\infty} \frac{\ln(x+1)}{x+1} dx = \left[ \frac{[\ln(x+1)]^2}{2} \right]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

8. 
$$\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$$

Let  $f(x) = \frac{\ln x}{\sqrt{x}}$ ,  $f'(x) = \frac{2 - \ln x}{2x^{3/2}}$ .

 $f$  is positive, continuous, and decreasing for  $x > e^2 \approx 7.4$ .

$$\int_2^{\infty} \frac{\ln x}{\sqrt{x}} dx = \left[ 2\sqrt{x}(\ln x - 2) \right]_2^{\infty} = \infty, \text{ diverges}$$

Hence, the series diverges by Theorem 9.10.

$$9. \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$$

$$\text{Let } f(x) = \frac{1}{\sqrt{x}(\sqrt{x}+1)}, f'(x) = -\frac{1+2\sqrt{x}}{2x^{3/2}(\sqrt{x}+1)^2} < 0.$$

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{1}{\sqrt{x}(\sqrt{x}+1)} dx = \left[ 2 \ln(\sqrt{x}+1) \right]_1^{\infty} = \infty, \text{ diverges}$$

Hence, the series diverges by Theorem 9.10.

$$11. \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$$

$$\text{Let } f(x) = \frac{1}{\sqrt{x+1}}, f'(x) = \frac{-1}{2(x+1)^{3/2}} < 0.$$

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{1}{\sqrt{x+1}} dx = \left[ 2\sqrt{x+1} \right]_1^{\infty} = \infty, \text{ diverges}$$

Hence, the series diverges by Theorem 9.10.

$$13. \sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

$$\text{Let } f(x) = \frac{\ln x}{x^2}, f'(x) = \frac{1-2\ln x}{x^3}.$$

$f$  is positive, continuous, and decreasing for  $x > e^{1/2} \approx 1.6$ .

$$\int_1^{\infty} \frac{\ln x}{x^2} dx = \left[ -\frac{(\ln x + 1)}{x} \right]_1^{\infty} = 1, \text{ converges}$$

Hence, the series converges by Theorem 9.10.

$$15. \sum_{n=1}^{\infty} \frac{\arctan n}{n^2 + 1}$$

$$\text{Let } f(x) = \frac{\arctan x}{x^2 + 1}, f'(x) = \frac{1 - 2x \arctan x}{(x^2 + 1)^2} < 0 \text{ for } x \geq 1.$$

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{\arctan x}{x^2 + 1} dx = \left[ \frac{(\arctan x)^2}{2} \right]_1^{\infty} = \frac{3\pi^2}{32}, \text{ converges}$$

Hence, the series converges by Theorem 9.10.

$$10. \sum_{n=1}^{\infty} \frac{n}{n^2 + 3}$$

$$\text{Let } f(x) = \frac{x}{x^2 + 3}.$$

$f(x)$  is positive, continuous, and decreasing for  $x \geq 2$  since

$$f'(x) = \frac{3 - x^2}{(x^2 + 3)^2} < 0 \text{ for } x \geq 2.$$

$$\int_1^{\infty} \frac{x}{x^2 + 3} dx = \left[ \ln \sqrt{x^2 + 3} \right]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

$$12. \sum_{n=2}^{\infty} \frac{\ln n}{n^3}$$

$$\text{Let } f(x) = \frac{\ln x}{x^3}, f'(x) = \frac{1 - 3 \ln x}{x^4}.$$

$f$  is positive, continuous, and decreasing for  $x > 2$ .

$$\int_2^{\infty} \frac{\ln x}{x^2} dx = \left[ -\frac{(2 \ln x + 1)}{4x^4} \right]_2^{\infty}$$

$$= \frac{2 \ln 2 + 1}{16}, \text{ converges}$$

Hence, the series converges by Theorem 9.10.

$$14. \sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

$$\text{Let } f(x) = \frac{1}{x\sqrt{\ln x}}, f'(x) = -\frac{2 \ln x + 1}{2x^2(\ln x)^{3/2}}.$$

$f$  is positive, continuous, and decreasing for  $x \geq 2$ .

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}} dx = \left[ 2\sqrt{\ln x} \right]_2^{\infty} = \infty, \text{ diverges}$$

Hence, the series diverges by Theorem 9.10.

$$16. \sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln n)}$$

$$\text{Let } f(x) = \frac{1}{x \ln x \ln(\ln x)},$$

$$f'(x) = \frac{-[(\ln x + 1) \ln(\ln x) + 1]}{x^2(\ln x)^2(\ln(\ln x))^2}.$$

$f$  is positive, continuous, and decreasing for  $x \geq 3$ .

$$\int_3^{\infty} \frac{1}{x \ln x \ln(\ln x)} dx = \left[ \ln(\ln(\ln x)) \right]_3^{\infty} = \infty, \text{ diverges}$$

Hence, the series diverges by Theorem 9.10.

$$17. \sum_{n=1}^{\infty} \frac{2n}{n^2 + 1}$$

$$\text{Let } f(x) = \frac{2x}{x^2 + 1}, f'(x) = 2 \frac{1 - x^2}{(x^2 + 1)^2} < 0 \text{ for } x > 1.$$

$f$  is positive, continuous, and decreasing for  $x > 1$ .

$$\int_1^{\infty} \frac{2x}{x^2 + 1} dx = \left[ \ln(x^2 + 1) \right]_1^{\infty} = \infty, \text{ diverges}$$

Hence, the series diverges by Theorem 9.10.

$$19. \sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k + c}$$

$$\text{Let } f(x) = \frac{x^{k-1}}{x^k + c}.$$

$f$  is positive, continuous, and decreasing for  $x > \sqrt[k]{c(k-1)}$  since

$$f'(x) = \frac{x^{k-2}[c(k-1) - x^k]}{(x^k + c)^2} < 0$$

for  $x > \sqrt[k]{c(k-1)}$ .

$$\int_1^{\infty} \frac{x^{k-1}}{x^k + c} dx = \left[ \frac{1}{k} \ln(x^k + c) \right]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

$$21. \text{ Let } f(x) = \frac{(-1)^x}{x}, f(n) = a_n.$$

The function  $f$  is not positive for  $x \geq 1$ .

$$23. \text{ Let } f(x) = \frac{2 + \sin x}{x}, f(n) = a_n.$$

The function  $f$  is not decreasing for  $x \geq 1$ .

$$25. \sum_{n=1}^{\infty} \frac{1}{n^3}$$

$$\text{Let } f(x) = \frac{1}{x^3}.$$

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{1}{x^3} dx = \left[ -\frac{1}{2x^2} \right]_1^{\infty} = \frac{1}{2}$$

Converges by Theorem 9.10

$$18. \sum_{n=1}^{\infty} \frac{n}{n^4 + 1}$$

$$\text{Let } f(x) = \frac{x}{x^4 + 1}, f'(x) = \frac{1 - 3x^4}{(x^4 + 1)^2} < 0 \text{ for } x > 1.$$

$f$  is positive, continuous, and decreasing for  $x > 1$ .

$$\int_1^{\infty} \frac{x}{x^4 + 1} dx = \left[ \frac{1}{2} \arctan(x^2) \right]_1^{\infty} = \frac{\pi}{8}, \text{ converges}$$

Hence, the series converges by Theorem 9.10.

$$20. \sum_{n=1}^{\infty} n^k e^{-n}$$

$$\text{Let } f(x) = \frac{x^k}{e^x}.$$

$f$  is positive, continuous, and decreasing for  $x > k$  since

$$f'(x) = \frac{x^{k-1}(k-x)}{e^x} < 0 \text{ for } x > k.$$

We use integration by parts.

$$\begin{aligned} \int_1^{\infty} x^k e^{-x} dx &= \left[ -x^k e^{-x} \right]_1^{\infty} + k \int_1^{\infty} x^{k-1} e^{-x} dx \\ &= \frac{1}{e} + \frac{k}{e} + \frac{k(k-1)}{e} + \dots + \frac{k!}{e} \end{aligned}$$

Converges by Theorem 9.10

$$22. \text{ Let } f(x) = e^{-x} \cos x, f(n) = a_n.$$

The function  $f$  is not positive for  $x \geq 1$ .

$$24. \text{ Let } f(x) = \left( \frac{\sin x}{x} \right)^2, f(n) = a_n.$$

The function  $f$  is not decreasing for  $x \geq 1$ .

$$26. \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$$

$$\text{Let } f(x) = \frac{1}{x^{1/3}}.$$

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{1}{x^{1/3}} dx = \left[ \frac{3}{2} x^{2/3} \right]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

27. 
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

Let  $f(x) = \frac{1}{\sqrt{x}}$ ,  $f'(x) = \frac{-1}{2x^{3/2}} < 0$  for  $x \geq 1$ .

 $f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{1}{\sqrt{x}} dx = \left[ 2\sqrt{x} \right]_1^{\infty} = \infty, \text{ diverges}$$

Hence, the series diverges by Theorem 9.10.

28. 
$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Let  $f(x) = \frac{1}{x^2}$ ,  $f'(x) = \frac{-2}{x^3} < 0$  for  $x \geq 1$ .

 $f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{1}{x^2} dx = \left[ -\frac{1}{x} \right]_1^{\infty} = 1, \text{ converges}$$

Hence, the series converges by Theorem 9.10.

29. 
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/5}}$$

Divergent  $p$ -series with  $p = \frac{1}{5} < 1$ 

30. 
$$\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$$

Convergent  $p$ -series with  $p = \frac{5}{3} > 1$ 

31. 
$$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

Divergent  $p$ -series with  $p = \frac{1}{2} < 1$ 

32. 
$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Convergent  $p$ -series with  $p = 2 > 1$ 

33. 
$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

Convergent  $p$ -series with  $p = \frac{3}{2} > 1$ 

34. 
$$\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$$

Divergent  $p$ -series with  $p = \frac{2}{3} < 1$ 

35. 
$$\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$$

Convergent  $p$ -series with  $p = 1.04 > 1$ 

36. 
$$\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$$

Convergent  $p$ -series with  $p = \pi > 1$ 

37. 
$$\sum_{n=1}^{\infty} \frac{2}{n^{3/4}}$$

$S_1 = 2$

$S_2 \approx 3.1892$

$S_3 \approx 4.0666$

$S_4 \approx 4.7740$

Matches (c), diverges

38. 
$$\sum_{n=1}^{\infty} \frac{2}{n}$$

$S_1 = 2$

$S_2 = 3$

$S_3 \approx 3.6667$

$S_4 \approx 4.1667$

Matches (f), diverges

39. 
$$\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^{\pi}}} = \sum_{n=1}^{\infty} \frac{2}{n^{\pi/2}}$$

$S_1 = 2$

$S_2 \approx 2.6732$

$S_3 \approx 3.0293$

$S_4 \approx 3.2560$

Matches (b), converges

40. 
$$\sum_{n=1}^{\infty} \frac{2}{n^{2/5}}$$

$S_1 = 2$

$S_2 \approx 3.5157$

$S_3 \approx 4.8045$

Matches (a), diverges

**Note:** The partial sums for 39 and 41 are very similar because  $\pi/2 \approx 3/2$ .

41. 
$$\sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{2}{n^{3/2}}$$

$S_1 = 2$

$S_2 \approx 2.7071$

$S_3 \approx 3.0920$

$S_4 \approx 3.3420$

Matches (d), converges

42. 
$$\sum_{n=1}^{\infty} \frac{2}{n^2}$$

$S_1 = 2$

$S_2 = 2.5$

$S_3 \approx 2.7222$

Matches (e), converges

**Note:** The partial sums for 39 and 41 are very similar because  $\pi/2 \approx 3/2$ .



43. (a)

|       |        |      |      |      |      |
|-------|--------|------|------|------|------|
| $n$   | 5      | 10   | 20   | 50   | 100  |
| $S_n$ | 3.7488 | 3.75 | 3.75 | 3.75 | 3.75 |

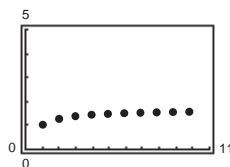
The partial sums approach the sum 3.75 very rapidly.



(b)

|       |        |        |        |        |       |
|-------|--------|--------|--------|--------|-------|
| $n$   | 5      | 10     | 20     | 50     | 100   |
| $S_n$ | 1.4636 | 1.5498 | 1.5962 | 1.6251 | 1.635 |

The partial sums approach the sum  $\pi^2/6 \approx 1.6449$  slower than the series in part (a).



44.  $\sum_{n=1}^N \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{N} > M$

(a)

|     |   |    |     |      |
|-----|---|----|-----|------|
| $M$ | 2 | 4  | 6   | 8    |
| $N$ | 4 | 31 | 227 | 1674 |

(b) No. Since the terms are decreasing (approaching zero), more and more terms are required to increase the partial sum by 2.

45. Let  $f$  be positive, continuous, and decreasing for  $x \geq 1$  and  $a_n = f(n)$ . Then,

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x) dx$$

either both converge or both diverge (Theorem 9.10). See Example 1, page 618.

46. A series of the form  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  is a  $p$ -series,  $p > 0$ .

The  $p$ -series converges if  $p > 1$  and diverges if  $0 < p \leq 1$ .

47. Your friend is not correct. The series

$$\sum_{n=10,000}^{\infty} \frac{1}{n} = \frac{1}{10,000} + \frac{1}{10,001} + \dots$$

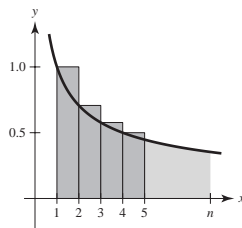
is the harmonic series, starting with the 10,000<sup>th</sup> term, and hence diverges.

48. No. Theorem 9.9 says that if the series converges, then the terms  $a_n$  tend to zero. Some of the series in Exercises 37–42 converge because the terms tend to 0 very rapidly.

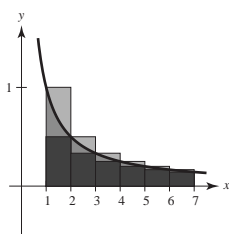
49. The area under the rectangles is greater than the area under the graph of  $y = 1/\sqrt{x}$ ,  $x \geq 1$ .

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{1}{\sqrt{x}} dx$$

Since  $\int_1^{\infty} \frac{1}{\sqrt{x}} dx = [2\sqrt{x}]_1^{\infty} = \infty$  diverges, the series  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$  diverges.



50.  $\sum_{n=1}^6 a_n \geq \int_1^7 f(x) dx \geq \sum_{n=2}^7 a_n$



51.  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

If  $p = 1$ , then the series diverges by the Integral Test. If  $p \neq 1$ ,

$$\int_2^{\infty} \frac{1}{x(\ln x)^p} dx = \int_2^{\infty} (\ln x)^{-p} \frac{1}{x} dx = \left[ \frac{(\ln x)^{-p+1}}{-p+1} \right]_2^{\infty}$$

Converges for  $-p + 1 < 0$  or  $p > 1$

$$52. \sum_{n=2}^{\infty} \frac{\ln n}{n^p}$$

If  $p = 1$ , then the series diverges by the Integral Test. If  $p \neq 1$ ,

$$\int_2^{\infty} \frac{\ln x}{x^p} dx = \int_2^{\infty} x^{-p} \ln x dx = \left[ \frac{x^{-p+1}}{(-p+1)^2} [-1 + (-p+1) \ln x] \right]_2^{\infty}. \quad (\text{Use integration by parts.})$$

Converges for  $-p + 1 < 0$  or  $p > 1$

$$53. \sum_{n=1}^{\infty} \frac{n}{(1+n^2)^p}$$

If  $p = 1$ ,  $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$  diverges (see Example 1). Let

$$f(x) = \frac{x}{(1+x^2)^p}, \quad p \neq 1$$

$$f'(x) = \frac{1 - (2p-1)x^2}{(1+x^2)^{p+1}}.$$

For a fixed  $p > 0$ ,  $p \neq 1$ ,  $f'(x)$  is eventually negative.  $f$  is positive, continuous, and eventually decreasing.

$$\int_1^{\infty} \frac{x}{(1+x^2)^p} dx = \left[ \frac{1}{(x^2+1)^{p-1}(2-2p)} \right]_1^{\infty}$$

For  $p > 1$ , this integral converges. For  $0 < p < 1$ , it diverges.

$$54. \sum_{n=1}^{\infty} n(1+n^2)^p$$

Since  $p > 0$ , the series diverges for all values of  $p$ .

$$55. \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges for } p > 1.$$

Hence,  $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ , diverges.

$$56. \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges for } p > 1.$$

Hence,  $\sum_{n=2}^{\infty} \frac{1}{n \sqrt[3]{(\ln n)^2}} = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{2/3}}$ , diverges.

$$57. \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges for } p > 1.$$

Hence,  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ , converges.

$$58. \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges for } p > 1.$$

Hence,  $\sum_{n=2}^{\infty} \frac{1}{n \ln(n^2)} = \sum_{n=2}^{\infty} \frac{1}{2n \ln n}$   
 $= \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ , diverges

59. Since  $f$  is positive, continuous, and decreasing for  $x \geq 1$  and  $a_n = f(n)$ , we have,

$$R_N = S - S_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^N a_n = \sum_{n=N+1}^{\infty} a_n > 0.$$

Also,

$$R_N = S - S_N = \sum_{n=N+1}^{\infty} a_n \leq a_{N+1} + \int_{N+1}^{\infty} f(x) dx$$

$$\leq \int_N^{\infty} f(x) dx.$$

Thus,  $0 \leq R_N \leq \int_N^{\infty} f(x) dx$ .

60. From Exercise 59, we have:

$$0 \leq S - S_N \leq \int_N^{\infty} f(x) dx$$

$$S_N \leq S \leq S_N + \int_N^{\infty} f(x) dx$$

$$\sum_{n=1}^N a_n \leq S \leq \sum_{n=1}^N a_n + \int_N^{\infty} f(x) dx$$

62.  $S_4 = 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} \approx 1.0363$

$$R_4 \leq \int_4^{\infty} \frac{1}{x^5} dx = \left[ -\frac{1}{4x^4} \right]_4^{\infty} \approx 0.0010$$

$$1.0363 \leq \sum_{n=1}^{\infty} \frac{1}{n^5} \leq 1.0363 + 0.0010 = 1.0373$$

63.  $S_{10} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \frac{1}{37} + \frac{1}{50} + \frac{1}{65} + \frac{1}{82} + \frac{1}{101} \approx 0.9818$

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^2 + 1} dx = \left[ \arctan x \right]_{10}^{\infty} = \frac{\pi}{2} - \arctan 10 \approx 0.0997$$

$$0.9818 \leq \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \leq 0.9818 + 0.0997 = 1.0815$$

64.  $S_{10} = \frac{1}{2(\ln 2)^3} + \frac{1}{3(\ln 3)^3} + \frac{1}{4(\ln 4)^3} + \cdots + \frac{1}{11(\ln 11)^3} \approx 1.9821$

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{(x+1)[\ln(x+1)]^3} dx = \left[ -\frac{1}{2[\ln(x+1)]^2} \right]_{10}^{\infty} = \frac{1}{2(\ln 11)^2} \approx 0.0870$$

$$1.9821 \leq \sum_{n=1}^{\infty} \frac{1}{(n+1)[\ln(n+1)]^3} \leq 1.9821 + 0.0870 = 2.0691$$

65.  $S_4 = \frac{1}{e} + \frac{2}{e^4} + \frac{3}{e^9} + \frac{4}{e^{16}} \approx 0.4049$

$$R_4 \leq \int_4^{\infty} x e^{-x^2} dx = \left[ -\frac{1}{2} e^{-x^2} \right]_4^{\infty} = \frac{e^{-16}}{2} \approx 5.6 \times 10^{-8}$$

$$0.4049 \leq \sum_{n=1}^{\infty} n e^{-n^2} \leq 0.4049 + 5.6 \times 10^{-8}$$

61.  $S_6 = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} \approx 1.0811$

$$R_6 \leq \int_6^{\infty} \frac{1}{x^4} dx = \left[ -\frac{1}{3x^3} \right]_6^{\infty} \approx 0.0015$$

$$1.0811 \leq \sum_{n=1}^{\infty} \frac{1}{n^4} \leq 1.0811 + 0.0015 = 1.0826$$

66.  $S_4 = \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \frac{1}{e^4} \approx 0.5713$

$$R_4 \leq \int_4^{\infty} e^{-x} dx = \left[ -e^{-x} \right]_4^{\infty} \approx 0.0183$$

$$0.5713 \leq \sum_{n=0}^{\infty} e^{-n} \leq 0.5713 + 0.0183 = 0.5896$$

67.  $0 \leq R_N \leq \int_N^{\infty} \frac{1}{x^4} dx = \left[ -\frac{1}{3x^3} \right]_N^{\infty} = \frac{1}{3N^3} < 0.001$

$$\frac{1}{N^3} < 0.003$$

$$N^3 > 333.33$$

$$N > 6.93$$

$$N \geq 7$$

68.  $0 \leq R_N \leq \int_N^{\infty} \frac{1}{x^{3/2}} dx = \left[ -\frac{2}{x^{1/2}} \right]_N^{\infty} = \frac{2}{\sqrt{N}} < 0.001$

$$N^{-1/2} < 0.0005$$

$$\sqrt{N} > 2000$$

$$N \geq 4,000,000$$

$$69. R_N \leq \int_N^\infty e^{-5x} dx = \left[ -\frac{1}{5} e^{-5x} \right]_N^\infty = \frac{e^{-5N}}{5} < 0.001$$

$$\frac{1}{e^{5N}} < 0.005$$

$$e^{5N} > 200$$

$$5N > \ln 200$$

$$N > \frac{\ln 200}{5}$$

$$N > 1.0597$$

$$N \geq 2$$

$$70. R_N \leq \int_N^\infty e^{-x/2} dx = \left[ -2e^{-x/2} \right]_N^\infty = \frac{2}{e^{N/2}} < 0.001$$

$$\frac{2}{e^{N/2}} < 0.001$$

$$e^{N/2} > 2000$$

$$\frac{N}{2} > \ln 2000$$

$$N > 2 \ln 2000 \approx 15.2$$

$$N \geq 16$$

$$71. R_N \leq \int_N^\infty \frac{1}{x^2 + 1} dx = \left[ \arctan x \right]_N^\infty \\ = \frac{\pi}{2} - \arctan N < 0.001$$

$$-\arctan N < -1.5698$$

$$\arctan N > 1.5698$$

$$N > \tan 1.5698$$

$$N \geq 1004$$

$$72. R_n \leq \int_N^\infty \frac{2}{x^2 + 5} dx = 2 \left[ \frac{1}{\sqrt{5}} \arctan \left( \frac{x}{\sqrt{5}} \right) \right]_N^\infty \\ = \frac{2}{\sqrt{5}} \left( \frac{\pi}{2} - \arctan \left( \frac{N}{\sqrt{5}} \right) \right) < 0.001$$

$$\frac{\pi}{2} - \arctan \left( \frac{N}{\sqrt{5}} \right) < 0.001118$$

$$1.56968 < \arctan \left( \frac{N}{\sqrt{5}} \right)$$

$$\frac{N}{\sqrt{5}} > \tan 1.56968$$

$$N \geq 2004$$

73. (a)  $\sum_{n=2}^{\infty} \frac{1}{n^{1.1}}$ . This is a convergent  $p$ -series with  $p = 1.1 > 1$ .  $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$  is a divergent series. Use the Integral Test.

$f(x) = \frac{1}{x \ln x}$  is positive, continuous, and decreasing for  $x \geq 2$ .

$$\int_2^\infty \frac{1}{x \ln x} dx = \left[ \ln |\ln x| \right]_2^\infty = \infty$$

$$(b) \sum_{n=2}^6 \frac{1}{n^{1.1}} = \frac{1}{2^{1.1}} + \frac{1}{3^{1.1}} + \frac{1}{4^{1.1}} + \frac{1}{5^{1.1}} + \frac{1}{6^{1.1}} \approx 0.4665 + 0.2987 + 0.2176 + 0.1703 + 0.1393$$

$$\sum_{n=2}^6 \frac{1}{n \ln n} = \frac{1}{2 \ln 2} + \frac{1}{3 \ln 3} + \frac{1}{4 \ln 4} + \frac{1}{5 \ln 5} + \frac{1}{6 \ln 6} \approx 0.7213 + 0.3034 + 0.1803 + 0.1243 + 0.0930$$

For  $n \geq 4$ , the terms of the convergent series **seem** to be larger than those of the divergent series!

$$(c) \frac{1}{n^{1.1}} < \frac{1}{n \ln n}$$

$$n \ln n < n^{1.1}$$

$$\ln n < n^{0.1}$$

This inequality holds when  $n \geq 3.5 \times 10^{15}$ . Or,  $n > e^{40}$ . Then  $\ln e^{40} = 40 < (e^{40})^{0.1} = e^4 \approx 55$ .

$$74. (a) \int_{10}^\infty \frac{1}{x^p} dx = \left[ \frac{x^{-p+1}}{-p+1} \right]_{10}^\infty = \frac{1}{(p-1)10^{p-1}}, \quad p > 1$$

$$(b) f(x) = \frac{1}{x^p}$$

(c) The horizontal asymptote is  $y = 0$ . As  $n$  increases, the error decreases.

$$R_{10}(p) = \sum_{n=11}^{\infty} \frac{1}{n^p}$$

$\leq$  Area under the graph of  $f$  over the interval  $[10, \infty)$

75. (a) Let  $f(x) = 1/x$ .  $f$  is positive, continuous, and decreasing on  $[1, \infty)$ .

$$S_n - 1 \leq \int_1^n \frac{1}{x} dx$$

$$S_n - 1 \leq \ln n$$

Hence,  $S_n \leq 1 + \ln n$ . Similarly,

$$S_n \geq \int_1^{n+1} \frac{1}{x} dx = \ln(n+1).$$

Thus,  $\ln(n+1) \leq S_n \leq 1 + \ln n$ .

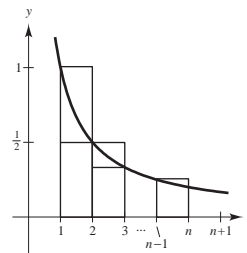
(b) Since  $\ln(n+1) \leq S_n \leq 1 + \ln n$ , we have  $\ln(n+1) - \ln n \leq S_n - \ln n \leq 1$ . Also, since  $\ln x$  is an increasing function,  $\ln(n+1) - \ln n > 0$  for  $n \geq 1$ . Thus,  $0 \leq S_n - \ln n \leq 1$  and the sequence  $\{a_n\}$  is bounded.

$$(c) a_n - a_{n+1} = [S_n - \ln n] - [S_{n+1} - \ln(n+1)] = \int_n^{n+1} \frac{1}{x} dx - \frac{1}{n+1} \geq 0$$

Thus,  $a_n \geq a_{n+1}$  and the sequence is decreasing.

(d) Since the sequence is bounded and monotonic, it converges to a limit,  $\gamma$ .

$$(e) a_{100} = S_{100} - \ln 100 \approx 0.5822 \text{ (Actually } \gamma \approx 0.577216.)$$



$$\begin{aligned} 76. \sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) &= \sum_{n=2}^{\infty} \ln\left(\frac{n^2 - 1}{n^2}\right) = \sum_{n=2}^{\infty} \ln \frac{(n+1)(n-1)}{n^2} = \sum_{n=2}^{\infty} [\ln(n+1) + \ln(n-1) - 2 \ln n] \\ &= \ln 3 + \ln 1 - 2 \ln 2 + (\ln 4 + \ln 2 - 2 \ln 3) + (\ln 5 + \ln 3 - 2 \ln 4) + (\ln 6 + \ln 4 - 2 \ln 5) \\ &\quad + (\ln 7 + \ln 5 - 2 \ln 6) + (\ln 8 + \ln 6 - 2 \ln 7) + (\ln 9 + \ln 7 - 2 \ln 8) + \dots = -\ln 2 \end{aligned}$$

$$77. \sum_{n=2}^{\infty} x^{\ln n}$$

$$(a) x = 1: \sum_{n=2}^{\infty} 1^{\ln n} = \sum_{n=2}^{\infty} 1, \text{ diverges}$$

$$(b) x = \frac{1}{e}: \sum_{n=2}^{\infty} \left(\frac{1}{e}\right)^{\ln n} = \sum_{n=2}^{\infty} e^{-\ln n} = \sum_{n=2}^{\infty} \frac{1}{n}, \text{ diverges}$$

(c) Let  $x$  be given,  $x > 0$ . Put  $x = e^{-p} \Leftrightarrow \ln x = -p$ .

$$\sum_{n=2}^{\infty} x^{\ln n} = \sum_{n=2}^{\infty} e^{-p \ln n} = \sum_{n=2}^{\infty} n^{-p} = \sum_{n=2}^{\infty} \frac{1}{n^p}$$

This series converges for  $p > 1 \Rightarrow x < \frac{1}{e}$ .

$$78. \xi(x) = \sum_{n=1}^{\infty} n^{-x} = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

Converges for  $x > 1$  by Theorem 9.11

$$79. \sum_{n=1}^{\infty} \frac{1}{2n-1}$$

$$\text{Let } f(x) = \frac{1}{2x-1}.$$

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{1}{2x-1} dx = \left[ \ln \sqrt{2x-1} \right]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

$$80. \sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$$

$$\text{Let } f(x) = \frac{1}{x\sqrt{x^2-1}}.$$

$f$  is positive, continuous, and decreasing for  $x \geq 2$ .

$$\int_2^{\infty} \frac{1}{x\sqrt{x^2-1}} dx = \left[ \operatorname{arcsec} x \right]_2^{\infty} = \frac{\pi}{2} - \frac{\pi}{3}$$

Converges by Theorem 9.10

$$81. \sum_{n=1}^{\infty} \frac{1}{n\sqrt[4]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$$

$p$ -series with  $p = \frac{5}{4}$

Converges by Theorem 9.11

$$82. 3 \sum_{n=1}^{\infty} \frac{1}{n^{0.95}}$$

$p$ -series with  $p = 0.95$

Diverges by Theorem 9.11

$$83. \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$$

Geometric series with  $r = \frac{2}{3}$

Converges by Theorem 9.6

$$84. \sum_{n=0}^{\infty} (1.075)^n$$

Geometric series with  $r = 1.075$

Diverges by Theorem 9.6

$$85. \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2 + 1}}$$

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2 + 1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + (1/n^2)}} = 1 \neq 0$$

Diverges by Theorem 9.9

$$86. \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n^3}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Since these are both convergent  $p$ -series, the difference is convergent.

$$87. \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \neq 0$$

Fails  $n$ th-Term Test

Diverges by Theorem 9.9

$$88. \sum_{n=2}^{\infty} \ln(n)$$

$$\lim_{n \rightarrow \infty} \ln(n) = \infty$$

Diverges by Theorem 9.9

$$89. \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$$

$$\text{Let } f(x) = \frac{1}{x(\ln x)^3}.$$

$f$  is positive, continuous and decreasing for  $x \geq 2$ .

$$\int_2^{\infty} \frac{1}{x(\ln x)^3} dx = \int_2^{\infty} (\ln x)^{-3} \frac{1}{x} dx = \left[ \frac{(\ln x)^{-2}}{-2} \right]_2^{\infty} = \left[ -\frac{1}{2(\ln x)^2} \right]_2^{\infty} = \frac{1}{2(\ln 2)^2}$$

Converges by Theorem 9.10. See Exercise 51.

$$90. \sum_{n=2}^{\infty} \frac{\ln n}{n^3}$$

$$\text{Let } f(x) = \frac{\ln x}{x^3}.$$

$f$  is positive, continuous, and decreasing for  $x \geq 2$  since  $f'(x) = \frac{1 - 3 \ln x}{x^4} < 0$  for  $x \geq 2$ .

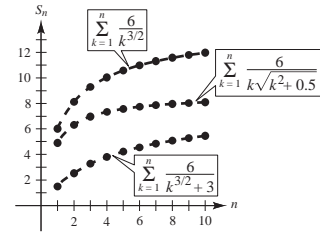
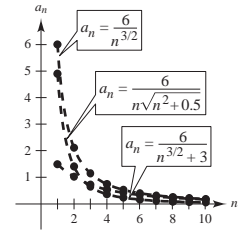
$$\int_2^{\infty} \frac{\ln x}{x^3} dx = \left[ -\frac{\ln x}{2x^2} \right]_2^{\infty} + \frac{1}{2} \int_2^{\infty} \frac{1}{x^3} dx = \frac{\ln 2}{8} + \left[ -\frac{1}{4x^2} \right]_2^{\infty} = \frac{\ln 2}{8} + \frac{1}{16} \quad (\text{Use integration by parts.})$$

Converges by Theorem 9.10. See Exercise 30.

### Section 9.4 Comparisons of Series

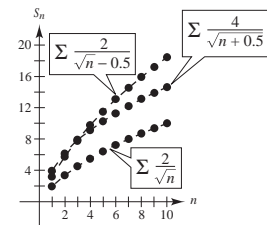
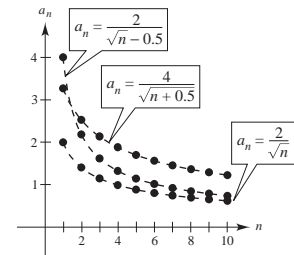
1. (a)  $\sum_{n=1}^{\infty} \frac{6}{n^{3/2}} = \frac{6}{1} + \frac{6}{2^{3/2}} + \dots; S_1 = 6$   
 $\sum_{n=1}^{\infty} \frac{6}{n^{3/2} + 3} = \frac{6}{4} + \frac{6}{2^{3/2} + 3} + \dots; S_1 = \frac{3}{2}$   
 $\sum_{n=1}^{\infty} \frac{6}{n\sqrt{n^2 + 0.5}} = \frac{6}{1\sqrt{1.5}} + \frac{6}{2\sqrt{4.5}} + \dots; S_1 = \frac{6}{\sqrt{1.5}} \approx 4.9$

- (b) The first series is a  $p$ -series. It converges ( $p = \frac{3}{2} > 1$ ).  
 (c) The magnitude of the terms of the other two series are less than the corresponding terms at the convergent  $p$ -series. Hence, the other two series converge.  
 (d) The smaller the magnitude of the terms, the smaller the magnitude of the terms of the sequence of partial sums.



2. (a)  $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}} = 2 + \frac{2}{\sqrt{2}} + \dots; S_1 = 2$   
 $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n} - 0.5} = \frac{2}{0.5} + \frac{2}{\sqrt{2} - 0.5} + \dots; S_1 = 4$   
 $\sum_{n=1}^{\infty} \frac{4}{\sqrt{n} + 0.5} = \frac{4}{\sqrt{1.5}} + \frac{4}{\sqrt{2.5}} + \dots; S_1 \approx 3.3$

- (b) The first series is a  $p$ -series. It diverges ( $p = \frac{1}{2} < 1$ ).  
 (c) The magnitude of the terms of the other two series are greater than the corresponding terms of the divergent  $p$ -series. Hence, the other two series diverge.  
 (d) The larger the magnitude of the terms, the larger the magnitude of the terms of the sequence of partial sums.



3.  $0 < \frac{1}{n^2 + 1} < \frac{1}{n^2}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

converges by comparison with the convergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

4.  $\frac{1}{3n^2 + 2} < \frac{1}{3n^2}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{3n^2 + 2}$$

converges by comparison with the convergent  $p$ -series

$$\frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

5.  $\frac{1}{n-1} > \frac{1}{n} > 0$  for  $n \geq 2$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{n-1}$$

diverges by comparison with the divergent  $p$ -series

$$\sum_{n=2}^{\infty} \frac{1}{n}$$

$$6. \frac{1}{\sqrt{n-1}} > \frac{1}{\sqrt{n}} \text{ for } n \geq 2$$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$$

diverges by comparison with the divergent  $p$ -series

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}.$$

$$7. 0 < \frac{1}{3^n + 1} < \frac{1}{3^n}$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{3^n + 1}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n.$$

$$8. \frac{3^n}{4^n + 5} < \left(\frac{3}{4}\right)^n$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{3^n}{4^n + 5}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n.$$

$$9. \text{ For } n \geq 3, \frac{\ln n}{n+1} > \frac{1}{n+1} > 0.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{\ln n}{n+1}$$

diverges by comparison with the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n+1}.$$

**Note:**  $\sum_{n=1}^{\infty} \frac{1}{n+1}$  diverges by the Integral Test.

$$10. \frac{1}{\sqrt{n^3+1}} < \frac{1}{n^{3/2}}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$$

converges by comparison with the convergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

$$11. \text{ For } n > 3, \frac{1}{n^2} > \frac{1}{n!} > 0.$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

converges by comparison with the convergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$12. \frac{1}{4\sqrt[3]{n-1}} > \frac{1}{4\sqrt[4]{n}}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n-1}}$$

diverges by comparison with the divergent  $p$ -series

$$\frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}.$$

$$13. 0 < \frac{1}{e^{n^2}} \leq \frac{1}{e^n}$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n.$$

$$14. \frac{4^n}{3^n - 1} > \frac{4^n}{3^n}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{4^n}{3^n - 1}$$

diverges by comparison with the divergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n.$$

$$15. \lim_{n \rightarrow \infty} \frac{n/(n^2+1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2+1} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

diverges by a limit comparison with the divergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$16. \lim_{n \rightarrow \infty} \frac{2/(3^n-5)}{1/3^n} = \lim_{n \rightarrow \infty} \frac{2 \cdot 3^n}{3^n-5} = 2$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2}{3^n-5}$$

converges by a limit comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n.$$



$$17. \lim_{n \rightarrow \infty} \frac{1/\sqrt{n^2+1}}{1/n} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+1}} = 1$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$$

diverges by a limit comparison with the divergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$19. \lim_{n \rightarrow \infty} \frac{2n^2-1}{3n^5+2n+1} = \lim_{n \rightarrow \infty} \frac{2n^5-n^3}{3n^5+2n+1} = \frac{2}{3}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^5+2n+1}$$

converges by a limit comparison with the convergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

$$21. \lim_{n \rightarrow \infty} \frac{n+3}{n(n+2)} = \lim_{n \rightarrow \infty} \frac{n^2+3n}{n^2+2n} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n+3}{n(n+2)}$$

diverges by a limit comparison with the divergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$23. \lim_{n \rightarrow \infty} \frac{1/(n\sqrt{n^2+1})}{1/n^2} = \lim_{n \rightarrow \infty} \frac{n^2}{n\sqrt{n^2+1}} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$$

converges by a limit comparison with the convergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$18. \lim_{n \rightarrow \infty} \frac{3/\sqrt{n^2-4}}{1/n} = \lim_{n \rightarrow \infty} \frac{3n}{\sqrt{n^2-4}} = 3$$

Therefore,

$$\sum_{n=3}^{\infty} \frac{3}{\sqrt{n^2-4}}$$

diverges by a limit comparison with the divergent harmonic series

$$\sum_{n=3}^{\infty} \frac{1}{n}.$$

$$20. \lim_{n \rightarrow \infty} \frac{5n-3}{n^2-2n+5} = \lim_{n \rightarrow \infty} \frac{5n^2-3n}{n^2-2n+5} = 5$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{5n-3}{n^2-2n+5}$$

diverges by a limit comparison with the divergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$22. \lim_{n \rightarrow \infty} \frac{1}{n(n^2+1)} = \lim_{n \rightarrow \infty} \frac{n^3}{n^3+n} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n(n^2+1)}$$

converges by a limit comparison with the convergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

$$24. \lim_{n \rightarrow \infty} \frac{n/[(n+1)2^{n-1}]}{1/(2^{n-1})} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}}$$

converges by a limit comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}.$$

$$25. \lim_{n \rightarrow \infty} \frac{(n^{k-1})/(n^k + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^k}{n^k + 1} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k + 1}$$

diverges by a limit comparison with the divergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$27. \lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = \lim_{n \rightarrow \infty} \frac{(-1/n^2) \cos(1/n)}{-1/n^2} \\ = \lim_{n \rightarrow \infty} \cos\left(\frac{1}{n}\right) = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

diverges by a limit comparison with the divergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$29. \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

Diverges

$p$ -series with  $p = \frac{1}{2}$

$$30. \sum_{n=0}^{\infty} 5\left(-\frac{1}{5}\right)^n$$

Converges

Geometric series with  $r = -\frac{1}{5}$

$$31. \sum_{n=1}^{\infty} \frac{1}{3^n + 2}$$

Converges

Direct comparison with  $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

$$32. \sum_{n=4}^{\infty} \frac{1}{3n^2 - 2n - 15}$$

Converges

Limit comparison with  $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$33. \sum_{n=1}^{\infty} \frac{n}{2n + 3}$$

Diverges;  $n$ th-Term Test

$$\lim_{n \rightarrow \infty} \frac{n}{2n + 3} = \frac{1}{2} \neq 0$$

$$34. \sum_{n=1}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+2} \right) = \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{5} \right) + \cdots = \frac{1}{2}$$

Converges; telescoping series

$$35. \sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2}$$

Converges; Integral Test

$$36. \sum_{n=1}^{\infty} \frac{3}{n(n+3)}$$

Converges; telescoping series

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+3} \right)$$

37.  $\lim_{n \rightarrow \infty} \frac{a_n}{1/n} = \lim_{n \rightarrow \infty} na_n$ . By given conditions  $\lim_{n \rightarrow \infty} na_n$  is finite and nonzero. Therefore,

$$\sum_{n=1}^{\infty} a_n$$

diverges by a limit comparison with the  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

39.  $\frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \frac{4}{17} + \frac{5}{26} + \cdots = \sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$ ,

which diverges since the degree of the numerator is only one less than the degree of the denominator.

41.  $\sum_{n=1}^{\infty} \frac{1}{n^3 + 1}$

converges since the degree of the numerator is three less than the degree of the denominator.

43.  $\lim_{n \rightarrow \infty} n \left( \frac{n^3}{5n^4 + 3} \right) = \lim_{n \rightarrow \infty} \frac{n^4}{5n^4 + 3} = \frac{1}{5} \neq 0$

Therefore,  $\sum_{n=1}^{\infty} \frac{n^3}{5n^4 + 3}$  diverges.

45.  $\frac{1}{200} + \frac{1}{400} + \frac{1}{600} + \cdots = \sum_{n=1}^{\infty} \frac{1}{200n}$

diverges, (harmonic)

47.  $\frac{1}{201} + \frac{1}{204} + \frac{1}{209} + \frac{1}{216} = \sum_{n=1}^{\infty} \frac{1}{200 + n^2}$

converges

49. Some series diverge or converge very slowly. You cannot decide convergence or divergence of a series by comparing the first few terms.

51. See Theorem 9.13, page 626. One example is

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}} \text{ diverges because } \lim_{n \rightarrow \infty} \frac{1/\sqrt{n-1}}{1/\sqrt{n}} = 1 \text{ and}$$

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges (} p\text{-series).}$$

38. If  $j < k - 1$ , then  $k - j > 1$ . The  $p$ -series with  $p = k - j$  converges and since

$$\lim_{n \rightarrow \infty} \frac{P(n)/Q(n)}{1/n^{k-j}} = L > 0, \text{ the series } \sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$$

converges by the Limit Comparison Test. Similarly, if  $j \geq k - 1$ , then  $k - j \leq 1$  which implies that

$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$$

diverges by the Limit Comparison Test.

40.  $\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \cdots = \sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$ ,

which converges since the degree of the numerator is two less than the degree of the denominator.

42.  $\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$

diverges since the degree of the numerator is only one less than the degree of the denominator.

44.  $\lim_{n \rightarrow \infty} \frac{n}{\ln n} = \lim_{n \rightarrow \infty} \frac{1}{1/n} = \lim_{n \rightarrow \infty} n = \infty \neq 0$

Therefore,  $\sum_{n=2}^{\infty} \frac{1}{\ln n}$  diverges.

46.  $\frac{1}{200} + \frac{1}{210} + \frac{1}{220} + \cdots = \sum_{n=0}^{\infty} \frac{1}{200 + 10n}$

diverges

48.  $\frac{1}{201} + \frac{1}{208} + \frac{1}{227} + \frac{1}{264} + \cdots = \sum_{n=1}^{\infty} \frac{1}{200 + n^3}$

converges

50. See Theorem 9.12, page 624. One example is

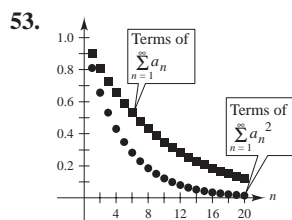
$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \text{ converges because } \frac{1}{n^2 + 1} < \frac{1}{n^2} \text{ and}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges (} p\text{-series).}$$

52. This is not correct. The beginning terms do not affect the convergence or divergence of a series. In fact,

$$\frac{1}{1000} + \frac{1}{1001} + \cdots = \sum_{n=1000}^{\infty} \frac{1}{n} \text{ diverges (harmonic)}$$

$$\text{and } 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges (} p\text{-series).}$$



For  $0 < a_n < 1$ ,  $0 < a_n^2 < a_n < 1$ . Hence, the lower terms are those of  $\sum a_n^2$ .

54. (a) 
$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 4n + 1}$$

converges since the degree of the numerator is two less than the degree of the denominator. (See Exercise 38.)

(c) 
$$\sum_{n=3}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} - S_2 \approx 0.1226$$

(b)

| $n$   | 5      | 10      | 20     | 50     | 100    |
|-------|--------|---------|--------|--------|--------|
| $S_n$ | 1.1839 | 1.02087 | 1.2212 | 1.2287 | 1.2312 |

(d) 
$$\sum_{n=10}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} - S_9 \approx 0.0277$$

55. False. Let  $a_n = \frac{1}{n^3}$  and  $b_n = \frac{1}{n^2}$ .  $0 < a_n \leq b_n$  and both  $\sum_{n=1}^{\infty} \frac{1}{n^3}$  and  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converge.

56. True

57. True

58. False. Let  $a_n = 1/n$ ,  $b_n = 1/n$ ,  $c_n = 1/n^2$ . Then,  $a_n \leq b_n + c_n$ , but  $\sum_{n=1}^{\infty} c_n$  converges.

59. True

60. False.  $\sum_{n=1}^{\infty} a_n$  could converge or diverge.

For example, let  $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ , which diverges.

$$0 < \frac{1}{n} < \frac{1}{\sqrt{n}} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges, but}$$

$$0 < \frac{1}{n^2} < \frac{1}{\sqrt{n}} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges.}$$

61. Since  $\sum_{n=1}^{\infty} b_n$  converges,  $\lim_{n \rightarrow \infty} b_n = 0$ . There exists  $N$  such that  $b_n < 1$  for  $n > N$ . Thus,  $a_n b_n < a_n$  for  $n > N$  and

$\sum_{n=1}^{\infty} a_n b_n$  converges by comparison to the convergent

series  $\sum_{i=1}^{\infty} a_n$ .

62. Since  $\sum_{n=1}^{\infty} a_n$  converges, then  $\sum_{n=1}^{\infty} a_n a_n = \sum_{n=1}^{\infty} a_n^2$  converges by Exercise 61.

63.  $\sum \frac{1}{n^2}$  and  $\sum \frac{1}{n^3}$  both converge, and hence so does

$$\sum \left( \frac{1}{n^2} \right) \left( \frac{1}{n^3} \right) = \sum \frac{1}{n^5}.$$

64.  $\sum \frac{1}{n^2}$  converge, and hence so does  $\sum \left( \frac{1}{n^2} \right)^2 = \sum \frac{1}{n^4}$ .

65. Suppose  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$  and  $\sum b_n$  converges.

From the definition of limit of a sequence, there exists  $M > 0$  such that

$$\left| \frac{a_n}{b_n} - 0 \right| < 1$$

whenever  $n > M$ . Hence,  $a_n < b_n$  for  $n > M$ . From the Comparison Test,  $\sum a_n$  converges.

66. Suppose  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$  and  $\sum b_n$  diverges. From the definition of limit of a sequence, there exists  $M > 0$  such that

$$\frac{a_n}{b_n} > 1$$

for  $n > M$ . Thus,  $a_n > b_n$  for  $n > M$ . By the Comparison Test,  $\sum a_n$  diverges.

67. (a) Let  $\sum a_n = \sum \frac{1}{(n+1)^3}$ , and  $\sum b_n = \sum \frac{1}{n^2}$ , converges.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1/[(n+1)^3]}{1/(n^2)} = \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^3} = 0$$

By Exercise 65,  $\sum_{n=1}^{\infty} \frac{1}{(n+1)^3}$  converges.

(b) Let  $\sum a_n = \sum \frac{1}{\sqrt{n}\pi^n}$ , and  $\sum b_n = \sum \frac{1}{\pi^n}$ , converges.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1/(\sqrt{n}\pi^n)}{1/(\pi^n)} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

By Exercise 65,  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\pi^n}$  converges.

68. (a) Let  $\sum a_n = \sum \frac{\ln n}{n}$ , and  $\sum b_n = \sum \frac{1}{n}$ , diverges.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{(\ln n)/n}{1/n} = \lim_{n \rightarrow \infty} \ln n = \infty$$

By Exercise 66,  $\sum_{n=1}^{\infty} \frac{\ln n}{n}$  diverges.

(b) Let  $\sum a_n = \sum \frac{1}{\ln n}$ , and  $\sum b_n = \sum \frac{1}{n}$ , diverges.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{\ln n} = \infty$$

By Exercise 66,  $\sum \frac{1}{\ln n}$  diverges.

69. Since  $\lim_{n \rightarrow \infty} a_n = 0$ , the terms of  $\sum \sin(a_n)$  are positive for sufficiently large  $n$ . Since

$$\lim_{n \rightarrow \infty} \frac{\sin(a_n)}{a_n} = 1 \text{ and } \sum a_n$$

converges, so does  $\sum \sin(a_n)$ .

$$\begin{aligned} 70. \sum_{n=1}^{\infty} \frac{1}{1+2+\cdots+n} &= \sum_{n=1}^{\infty} \frac{1}{[n(n+1)]/2} \\ &= \sum_{n=1}^{\infty} \frac{2}{n(n+1)} \end{aligned}$$

Since  $\sum 1/n^2$  converges, and

$$\lim_{n \rightarrow \infty} \frac{2/[n(n+1)]}{1/(n^2)} = \lim_{n \rightarrow \infty} \frac{2n^2}{n(n+1)} = 2,$$

$\sum \frac{1}{1+2+\cdots+n}$  converges.

71. The series diverges. For  $n > 1$ ,

$$n < 2^n$$

$$n^{1/n} < 2$$

$$\frac{1}{n^{1/n}} > \frac{1}{2}$$

$$\frac{1}{n^{(n+1)/n}} > \frac{1}{2n}$$

Since  $\sum \frac{1}{2n}$  diverges, so does  $\sum \frac{1}{n^{(n+1)/n}}$ .

72. Consider two cases:

If  $a_n \geq \frac{1}{2^{n+1}}$ , then  $a_n^{1/(n+1)} \geq \left(\frac{1}{2^{n+1}}\right)^{1/(n+1)} = \frac{1}{2}$ , and

$$a_n^{n/(n+1)} = \frac{a_n}{a_n^{1/(n+1)}} \leq 2a_n.$$

If  $a_n \leq \frac{1}{2^{n+1}}$ , then  $a_n^{n/(n+1)} \leq \left(\frac{1}{2^{n+1}}\right)^{n/(n+1)} = \frac{1}{2^n}$ , and

combining,  $a_n^{n/(n+1)} \leq 2a_n + \frac{1}{2^n}$ .

Since  $\sum_{n=1}^{\infty} \left(2a_n + \frac{1}{2^n}\right)$  converges, so does  $\sum_{n=1}^{\infty} a_n^{n/(n+1)}$  by the Comparison Test.

## Section 9.5 Alternating Series

1.  $\sum_{n=1}^{\infty} \frac{6}{n^2}$

$S_1 = 6$

$S_2 = 7.5$

$S_3 \approx 8.1667$

Matches (d).

2.  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}6}{n^2}$

$S_1 = 6$

$S_2 = 4.4$

$S_3 \approx 5.1667$

Matches (f).

3.  $\sum_{n=1}^{\infty} \frac{3}{n!}$

$S_1 = 3$

$S_2 = 4.5$

$S_3 = 5.0$

Matches (a).

4.  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}3}{n!}$

$S_1 = 3$

$S_2 = 1.5$

$S_3 = 2.0$

Matches (b).

5.  $\sum_{n=1}^{\infty} \frac{10}{n2^n}$

$S_1 = 5$

$S_2 = 6.25$

Matches (e).

6.  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}10}{n2^n}$

$S_1 = 5$

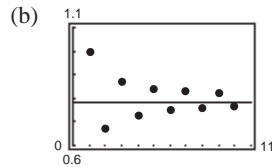
$S_2 = 3.75$

Matches (c).

7.  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4} \approx 0.7854$

(a)

| $n$   | 1 | 2      | 3      | 4      | 5      | 6      | 7      | 8      | 9      | 10     |
|-------|---|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $S_n$ | 1 | 0.6667 | 0.8667 | 0.7238 | 0.8349 | 0.7440 | 0.8209 | 0.7543 | 0.8131 | 0.7605 |



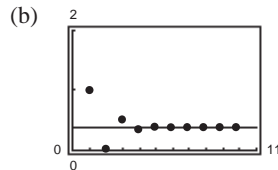
(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next term of the series.

8.  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} = \frac{1}{e} \approx 0.3679$

(a)

| $n$   | 1 | 2 | 3   | 4      | 5     | 6      | 7      | 8      | 9      | 10     |
|-------|---|---|-----|--------|-------|--------|--------|--------|--------|--------|
| $S_n$ | 1 | 0 | 0.5 | 0.3333 | 0.375 | 0.3667 | 0.3681 | 0.3679 | 0.3679 | 0.3679 |



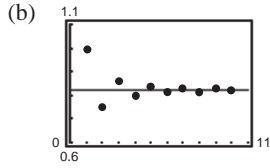
(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next series.

$$9. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12} \approx 0.8225$$

(a)

| $n$   | 1 | 2    | 3      | 4      | 5      | 6      | 7      | 8      | 9      | 10     |
|-------|---|------|--------|--------|--------|--------|--------|--------|--------|--------|
| $S_n$ | 1 | 0.75 | 0.8611 | 0.7986 | 0.8386 | 0.8108 | 0.8312 | 0.8156 | 0.8280 | 0.8180 |



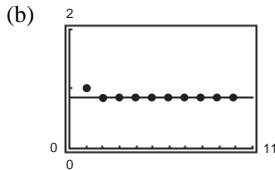
(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next term in the series.

$$10. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} = \sin(1) \approx 0.8415$$

(a)

| $n$   | 1 | 2      | 3      | 4      | 5      | 6      | 7      | 8      | 9      | 10     |
|-------|---|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $S_n$ | 1 | 0.8333 | 0.8417 | 0.8415 | 0.8415 | 0.8415 | 0.8415 | 0.8415 | 0.8415 | 0.8415 |



(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next series.

$$11. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

$$a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Converges by Theorem 9.14

$$12. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{2n-1}$$

$$\lim_{n \rightarrow \infty} \frac{n}{2n-1} = \frac{1}{2}$$

Diverges by the  $n$ th-Term Test

$$13. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

$$a_{n+1} = \frac{1}{2(n+1)-1} < \frac{1}{2n-1} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{2n-1} = 0$$

Converges by Theorem 9.14

$$14. \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$$

$$a_{n+1} = \frac{1}{\ln(n+2)} < \frac{1}{\ln(n+1)} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{\ln(n+1)} = 0$$

Converges by Theorem 9.14

$$15. \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2 + 1}$$

$$\lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} = 1. \text{ Thus, } \lim_{n \rightarrow \infty} a_n \neq 0.$$

Diverges by the  $n$ th-Term Test

$$17. \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

$$a_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

Converges by Theorem 9.14

$$19. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{\ln(n+1)}$$

$$\lim_{n \rightarrow \infty} \frac{n+1}{\ln(n+1)} = \lim_{n \rightarrow \infty} \frac{1}{1/(n+1)} = \lim_{n \rightarrow \infty} (n+1) = \infty$$

Diverges by the  $n$ th-Term Test

$$21. \sum_{n=1}^{\infty} \sin\left[\frac{(2n-1)\pi}{2}\right] = \sum_{n=1}^{\infty} (-1)^{n+1}$$

Diverges by the  $n$ th-Term Test

$$23. \sum_{n=1}^{\infty} \cos n\pi = \sum_{n=1}^{\infty} (-1)^n$$

Diverges by the  $n$ th-Term Test

$$26. \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

$$a_{n+1} = \frac{1}{(2n+3)!} < \frac{1}{(2n+1)!} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{(2n+1)!} = 0$$

Converges by Theorem 9.14

$$27. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+2}$$

$$a_{n+1} = \frac{\sqrt{n+1}}{(n+1)+2}$$

$$< \frac{\sqrt{n}}{n+2} \text{ for } n \geq 2$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n+2} = 0$$

Converges by Theorem 9.14

$$16. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{n^2+1}$$

$$a_{n+1} = \frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{n}{n^2+1} = 0$$

Converges by Theorem 9.14

$$18. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^2}{n^2+5}$$

$$\lim_{n \rightarrow \infty} \frac{n^2}{n^2+5} = 1$$

Diverges by  $n$ th-Term Test

$$20. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln(n+1)}{n+1}$$

$$a_{n+1} = \frac{\ln[(n+1)+1]}{(n+1)+1} < \frac{\ln(n+1)}{n+1} \text{ for } n \geq 2$$

$$\lim_{n \rightarrow \infty} \frac{\ln(n+1)}{n+1} = \lim_{n \rightarrow \infty} \frac{1/(n+1)}{1} = 0$$

Converges by Theorem 9.14

$$22. \sum_{n=1}^{\infty} \frac{1}{n} \sin\left[\frac{(2n-1)\pi}{2}\right] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Converges; (See Exercise 11.)

$$25. \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

$$a_{n+1} = \frac{1}{(n+1)!} < \frac{1}{n!} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n!} = 0$$

Converges by Theorem 9.14

$$28. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{\sqrt[3]{n}}$$

$$\lim_{n \rightarrow \infty} \frac{n^{1/2}}{n^{1/3}} = \lim_{n \rightarrow \infty} n^{1/6} = \infty$$

Diverges by the  $n$ th-Term Test



$$29. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$a_{n+1} = \frac{(n+1)!}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} = \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \cdot \frac{n+1}{2n+1} = a_n \left( \frac{n+1}{2n+1} \right) < a_n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \\ &= \lim_{n \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \\ &= \lim_{n \rightarrow \infty} 2 \left[ \frac{3}{3} \cdot \frac{4}{5} \cdot \frac{5}{7} \cdots \frac{n}{2n-3} \right] \cdot \frac{1}{2n-1} = 0 \end{aligned}$$

Converges by Theorem 9.14

$$30. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{1 \cdot 4 \cdot 7 \cdots (3n-2)}$$

$$a_{n+1} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{1 \cdot 4 \cdot 7 \cdots (3n-2)(3n+1)} = a_n \left( \frac{2n+1}{3n+1} \right) < a_n$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} 3 \left[ \frac{5}{4} \cdot \frac{7}{7} \cdot \frac{9}{10} \cdots \frac{2n-1}{3n-5} \right] \frac{1}{3n-2} = 0$$

Converges by Theorem 9.14

$$31. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2)}{e^n - e^{-n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2e^n)}{e^{2n} - 1}$$

Let  $f(x) = \frac{2e^x}{e^{2x} - 1}$ . Then

$$f'(x) = \frac{-2e^x(e^{2x} + 1)}{(e^{2x} - 1)^2} < 0.$$

Thus,  $f(x)$  is decreasing. Therefore,  $a_{n+1} < a_n$ , and

$$\lim_{n \rightarrow \infty} \frac{2e^n}{e^{2n} - 1} = \lim_{n \rightarrow \infty} \frac{2e^n}{2e^{2n}} = \lim_{n \rightarrow \infty} \frac{1}{e^n} = 0.$$

The series converges by Theorem 9.14.

$$32. \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n + e^{-n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2e^n)}{e^{2n} + 1}$$

Let  $f(x) = \frac{2e^x}{e^{2x} + 1}$ . Then

$$f'(x) = \frac{2e^{2x}(1 - e^{2x})}{(e^{2x} + 1)^2} < 0 \text{ for } x > 0.$$

Thus,  $f(x)$  is decreasing for  $x > 0$  which implies  $a_{n+1} < a_n$ .

$$\lim_{n \rightarrow \infty} \frac{2e^n}{e^{2n} + 1} = \lim_{n \rightarrow \infty} \frac{2e^n}{2e^{2n}} = \lim_{n \rightarrow \infty} \frac{1}{e^n} = 0$$

The series converges by Theorem 9.14.

$$33. S_6 = \sum_{n=1}^6 \frac{3(-1)^{n+1}}{n^2} = 2.4325$$

$$2.4325 - 0.0612 \leq S \leq 2.4325 + 0.0612$$

$$|R_6| = |S - S_6| \leq a_7 = \frac{3}{49} \approx 0.0612$$

$$2.3713 \leq S \leq 2.4937$$

$$35. S_6 = \sum_{n=0}^5 \frac{2(-1)^n}{n!} \approx 0.7333$$

$$0.7333 - 0.002778 \leq S \leq 0.7333 + 0.002778$$

$$|R_6| = |S - S_6| \leq a_7 = \frac{2}{6!} = 0.002778$$

$$0.7305 \leq S \leq 0.7361$$

$$34. S_6 = \sum_{n=1}^6 \frac{4(-1)^{n+1}}{\ln(n+1)} \approx 2.7067$$

$$|R_6| = |S - S_6| \leq a_7 = \frac{4}{\ln 8} \approx 1.9236$$

$$0.7831 \leq S \leq 4.6303$$

$$36. S_6 = \sum_{n=1}^6 \frac{(-1)^{n+1} n}{2^n} = 0.1875$$

$$|R_6| = |S - S_6| \leq a_7 = \frac{7}{2^7} \approx 0.05469$$

$$0.1328 \leq S \leq 0.2422$$

$$37. \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

(a) By Theorem 9.15,

$$|R_N| \leq a_{N+1} = \frac{1}{(N+1)!} < 0.001.$$

This inequality is valid when  $N = 6$ .

(b) We may approximate the series by

$$\sum_{n=0}^6 \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.368.$$

(7 terms. Note that the sum begins with  $n = 0$ .)

$$39. \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

(a) By Theorem 9.15,

$$|R_N| \leq a_{N+1} = \frac{1}{[2(N+1)+1]!} < 0.001.$$

This inequality is valid when  $N = 2$ .

(b) We may approximate the series by

$$\sum_{n=0}^2 \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{6} + \frac{1}{120} \approx 0.842.$$

(3 terms. Note that the sum begins with  $n = 0$ .)

$$41. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

(a) By Theorem 9.15,

$$|R_N| \leq a_{N+1} = \frac{1}{N+1} < 0.001.$$

This inequality is valid when  $N = 1000$ .

(b) We may approximate the series by

$$\sum_{n=1}^{1000} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1000} \approx 0.693.$$

(1000 terms)

$$43. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$$

By Theorem 9.15,

$$|R_N| \leq a_{N+1} = \frac{1}{(N+1)^3} < 0.001$$

$$\Rightarrow (N+1)^3 > 1000 \Rightarrow N+1 > 10.$$

Use 10 terms.

$$38. \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$$

(a) By Theorem 9.15,

$$|R_n| \leq a_{N+1} = \frac{1}{2^{N+1}(N+1)!} < 0.001.$$

This inequality is valid when  $N = 4$ .

(b) We may approximate the series by

$$\sum_{n=0}^4 \frac{(-1)^n}{2^n n!} = 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{348} \approx 0.607.$$

(5 terms. Note that the sum begins with  $n = 0$ .)

$$40. \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$$

(a) By Theorem 9.15,

$$|R_N| \leq a_{N+1} = \frac{1}{(2N+2)!} < 0.001.$$

This inequality is valid when  $N = 3$ .

(b) We may approximate the series by

$$\sum_{n=0}^3 \frac{(-1)^n}{(2n)!} = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} \approx 0.540.$$

(4 terms. Note that the sum begins with  $n = 0$ .)

$$42. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^n n}$$

(a) By Theorem 9.15,

$$|R_N| \leq a_{N+1} = \frac{1}{4^{N+1}(N+1)} < 0.001.$$

This inequality is valid when  $N = 3$ .

(b) We may approximate the series by

$$\sum_{n=1}^3 \frac{(-1)^{n+1}}{4^n n} = \frac{1}{4} - \frac{1}{32} + \frac{1}{192} \approx 0.224.$$

(3 terms)

$$44. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

By Theorem 9.15,

$$|R_N| \leq a_{N+1} = \frac{1}{(N+1)^2} < 0.001$$

$$\Rightarrow (N+1)^2 > 1000 \Rightarrow N = 31.$$

Use 31 terms.

$$45. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3 - 1}$$

By Theorem 9.15,

$$|R_N| \leq a_{N+1} = \frac{1}{2(N+1)^3 - 1} < 0.001.$$

This inequality is valid when  $N = 7$ . Use 7 terms.

$$47. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2}$$

$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$  converges by comparison to the  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Therefore, the given series converges absolutely.

$$49. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

The given series converges by the Alternating Series Test, but does not converge absolutely since

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is a divergent  $p$ -series. Therefore, the series converges conditionally.

$$51. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{(n+1)^2}$$

$$\lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} = 1$$

Therefore, the series diverges by the  $n$ th-Term Test.

$$53. \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$$

The given series converges by the Alternating Series Test but does not converge absolutely since the series

$\sum_{n=2}^{\infty} \frac{1}{\ln n}$  diverges by comparison to the harmonic series

$\sum_{n=1}^{\infty} \frac{1}{n}$ . Therefore, the series converges conditionally.

$$55. \sum_{n=2}^{\infty} \frac{(-1)^n n}{n^3 - 1}$$

$\sum_{n=2}^{\infty} \frac{n}{n^3 - 1}$  converges by a limit comparison to

the convergent  $p$ -series  $\sum_{n=2}^{\infty} \frac{1}{n^2}$ . Therefore, the given series converges absolutely.

$$46. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$$

By Theorem 9.15,  $|R_N| \leq a_{N+1} = \frac{1}{(N+1)^4} < 0.001$ .

This inequality is valid when  $N = 5$ .

$$48. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

The given series converges by the Alternating Series Test, but does not converge absolutely since the series

$$\sum_{n=1}^{\infty} \frac{1}{n+1}$$

diverges by the Integral Test. Therefore, the series converges conditionally.

$$50. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$$

$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$  which is a convergent  $p$ -series.

Therefore, the given series converges absolutely.

$$52. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n+3)}{n+10}$$

$$\lim_{n \rightarrow \infty} \frac{2n+3}{n+10} = 2$$

Therefore, the series diverges by the  $n$ th-Term Test.

$$54. \sum_{n=0}^{\infty} \frac{(-1)^n}{e^{n^2}}$$

$\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$  converges by a comparison to

the convergent geometric series  $\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$ . Therefore, the given series converges absolutely.

$$56. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1.5}}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$  is a convergent  $p$ -series.

Therefore, the given series converges absolutely.

$$57. \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$$

is convergent by comparison to the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

since

$$\frac{1}{(2n+1)!} < \frac{1}{2^n} \text{ for } n > 0.$$

Therefore, the given series converges absolutely.

$$59. \sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

The given series converges by the Alternating Series Test, but

$$\sum_{n=0}^{\infty} \frac{|\cos n\pi|}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

diverges by a limit comparison to the divergent harmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$\lim_{n \rightarrow \infty} \frac{|\cos n\pi|/(n+1)}{1/n} = 1$ , therefore the series converges conditionally.

$$61. \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$  is a convergent  $p$ -series.

Therefore, the given series converges absolutely.

63. An alternating series is a series whose terms alternate in sign. See Theorem 9.14.

65.  $\sum a_n$  is absolutely convergent if  $\sum |a_n|$  converges.  $\sum a_n$  is conditionally convergent if  $\sum |a_n|$  diverges, but  $\sum a_n$  converges.

67. False. Let  $a_n = \frac{(-1)^n}{n}$ .

Then  $\sum a_n$  converges and  $\sum (-a_n)$  converges.

But,  $\sum |a_n| = \sum \frac{1}{n}$  diverges.

$$58. \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$$

The given series converges by the Alternating Series Test, but

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}}$$

diverges by a limit comparison to the divergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

Therefore, the given series converges conditionally.

$$60. \sum_{n=1}^{\infty} (-1)^{n+1} \arctan n$$

$$\lim_{n \rightarrow \infty} \arctan n = \frac{\pi}{2} \neq 0$$

Therefore, the series diverges by the  $n$ th-Term Test.

$$62. \sum_{n=1}^{\infty} \frac{\sin[(2n-1)\pi/2]}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

The given series converges by the Alternating Series Test, but

$$\sum_{n=1}^{\infty} \left| \frac{\sin[(2n-1)\pi/2]}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

is a divergent  $p$ -series. Therefore, the series converges conditionally.

$$64. |S - S_n| = |R_n| \leq a_{n+1} \text{ (Theorem 9.15)}$$

66. (b). The partial sums alternate above and below the horizontal line representing the sum.

68. True. If  $\sum |a_n|$  converged, then so would  $\sum a_n$  by Theorem 9.16.

69. True.  $S_{100} = -1 + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{100}$

Since the next term  $-\frac{1}{101}$  is negative,  $S_{100}$  is an overestimate of the sum.

71.  $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^p}$

If  $p = 0$ , then  $\sum_{n=1}^{\infty} (-1)^n$  diverges.

If  $p < 0$ , then  $\sum_{n=1}^{\infty} (-1)^n n^{-p}$  diverges.

If  $p > 0$ , then  $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$  and

$$a_{n+1} = \frac{1}{(n+1)^p} < \frac{1}{n^p} = a_n.$$

Therefore, the series converges for  $p > 0$ .

73. Since

$$\sum_{n=1}^{\infty} |a_n|$$

converges we have  $\lim_{n \rightarrow \infty} |a_n| = 0$ . Thus, there must exist an  $N > 0$  such that  $|a_n| < 1$  for all  $n > N$  and it follows that  $a_n^2 \leq |a_n|$  for all  $n > N$ . Hence, by the Comparison Test,

$$\sum_{n=1}^{\infty} a_n^2$$

converges. Let  $a_n = 1/n$  to see that the converse is false.

74.  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$  converges, but  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges.

76. (a)  $\sum_{n=1}^{\infty} \frac{x^n}{n}$

converges absolutely (by comparison) for  $-1 < x < 1$ , since

$$\left| \frac{x^n}{n} \right| < |x^n| \text{ and } \sum x^n$$

is a convergent geometric series for  $-1 < x < 1$ .

(b) When  $x = -1$ , we have the convergent alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

When  $x = 1$ , we have the divergent harmonic series  $1/n$ . Therefore,

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \text{ converges conditionally for } x = -1.$$

70. False. Let

$$\sum a_n = \sum b_n = \sum \frac{(-1)^n}{\sqrt{n}}.$$

Then both converge by the Alternating Series Test. But,

$$\sum a_n b_n = \sum \frac{1}{n}, \text{ which diverges.}$$

72.  $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+p}$

Assume that  $n+p \neq 0$  so that  $a_n = 1/(n+p)$  are defined for all  $n$ . For all  $p$ ,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n+p} = 0$$

$$a_{n+1} = \frac{1}{n+1+p} < \frac{1}{n+p} < a_n.$$

Therefore, the series converges for all  $p$ .

75.  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges, hence so does  $\sum_{n=1}^{\infty} \frac{1}{n^4}$ .

77. (a) No, the series does not satisfy  $a_{n+1} \leq a_n$  for all  $n$ . For example,  $\frac{1}{9} < \frac{1}{8}$ .

(b) Yes, the series converges.

$$S_{2n} = \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{2^n} - \frac{1}{3^n}$$

$$= \left( \frac{1}{2} + \cdots + \frac{1}{2^n} \right) - \left( \frac{1}{3} + \cdots + \frac{1}{3^n} \right)$$

$$= \left( 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} \right) - \left( 1 + \frac{1}{3} + \cdots + \frac{1}{3^n} \right)$$

As  $n \rightarrow \infty$ ,

$$S_{2n} \rightarrow \frac{1}{1 - (1/2)} - \frac{1}{1 - (1/3)} = 2 - \frac{3}{2} = \frac{1}{2}.$$

78. (a) No, the series does not satisfy  $a_{n+1} \leq a_n$ :

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = 1 - \frac{1}{8} + \frac{1}{\sqrt{3}} - \frac{1}{64} + \cdots \text{ and}$$

$$\frac{1}{8} < \frac{1}{\sqrt{3}}.$$

(b) No, the series diverges because  $\sum \frac{1}{\sqrt{n}}$  diverges.

79.  $\sum_{n=1}^{\infty} \frac{10}{n^{3/2}} = 10 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ ,  
convergent  $p$ -series

80.  $\sum_{n=1}^{\infty} \frac{3}{n^2 + 5}$   
converges by limit comparison to  
convergent  $p$ -series  
 $\sum \frac{1}{n^2}$ .

81. Diverges by  $n$ th-Term Test

$$\lim_{n \rightarrow \infty} a_n = \infty$$

82. Converges by limit comparison to  
convergent geometric series  $\sum \frac{1}{2^n}$ .

83. Convergent geometric series  
 $(r = \frac{7}{8} < 1)$

84. Diverges by  $n$ th-Term Test

$$\lim_{n \rightarrow \infty} a_n = \frac{3}{2}$$

85. Convergent geometric series  
( $r = 1/\sqrt{e}$ ) or Integral Test

86. Converges (conditionally) by  
Alternating Series Test

87. Converges (absolutely) by  
Alternating Series Test

88. Diverges by comparison to  
Divergent Harmonic Series:

$$\frac{\ln n}{n} > \frac{1}{n} \text{ for } n \geq 3$$

89. The first term of the series is zero,  
not one. You cannot regroup series  
terms arbitrarily.

90. Rearranging the terms of an  
alternating series can change  
the sum.

91.  $s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

$$S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

$$(i) s_{4n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{1}{4n-1} - \frac{1}{4n}$$

$$\frac{1}{2}s_{2n} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \cdots + \frac{1}{4n-2} - \frac{1}{4n}$$

$$\text{Adding: } s_{4n} + \frac{1}{2}s_{2n} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} = S_{3n}$$

(ii)  $\lim_{n \rightarrow \infty} s_n = s$  (In fact,  $s = \ln 2$ .)

$$s \neq 0 \text{ since } s > \frac{1}{2}.$$

$$S = \lim_{n \rightarrow \infty} S_{3n} = s_{4n} + \frac{1}{2}s_{2n} = s + \frac{1}{2}s = \frac{3}{2}s$$

Thus,  $S \neq s$ .

## Section 9.6 The Ratio and Root Tests

$$1. \frac{(n+1)!}{(n-2)!} = \frac{(n+1)(n)(n-1)(n-2)!}{(n-2)!}$$

$$= (n+1)(n)(n-1)$$

$$2. \frac{(2k-2)!}{(2k)!} = \frac{(2k-2)!}{(2k)(2k-1)(2k-2)!}$$

$$= \frac{1}{(2k)(2k-1)}$$

3. Use the Principle of Mathematical Induction. When  $k = 1$ , the formula is valid since  $1 = \frac{(2(1))!}{2^1 \cdot 1!}$ . Assume that

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{(2n)!}{2^n n!}$$

and show that

$$1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1) = \frac{(2n+2)!}{2^{n+1}(n+1)!}$$

To do this, note that:

$$1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1) = [1 \cdot 3 \cdot 5 \cdots (2n-1)](2n+1)$$

$$= \frac{(2n)!}{2^n n!} \cdot (2n+1) \quad (\text{Induction hypothesis})$$

$$= \frac{(2n)!(2n+1)}{2^n n!} \cdot \frac{(2n+2)}{2(n+1)}$$

$$= \frac{(2n)!(2n+1)(2n+2)}{2^{n+1} n!(n+1)}$$

$$= \frac{(2n+2)!}{2^{n+1}(n+1)!}$$

The formula is valid for all  $n \geq 1$ .

4. Use the Principle of Mathematical Induction. When  $k = 3$ , the formula is valid since  $\frac{1}{1} = \frac{2^3 3!(3)(5)}{6!} = 1$ . Assume that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)} = \frac{2^n n!(2n-3)(2n-1)}{(2n)!}$$

and show that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)(2n-3)} = \frac{2^{n+1}(n+1)!(2n-1)(2n+1)}{(2n+2)!}$$

To do this, note that:

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)(2n-3)} = \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)} \cdot \frac{1}{(2n-3)}$$

$$= \frac{2^n n!(2n-3)(2n-1)}{(2n)!} \cdot \frac{1}{(2n-3)}$$

$$= \frac{2^n n!(2n-1)}{(2n)!} \cdot \frac{(2n+1)(2n+2)}{(2n+1)(2n+2)}$$

$$= \frac{2^n (2)(n+1)n!(2n-1)(2n+1)}{(2n)!(2n+1)(2n+2)}$$

$$= \frac{2^{n+1}(n+1)!(2n-1)(2n+1)}{(2n+2)!}$$

The formula is valid for all  $n \geq 3$ .

$$5. \sum_{n=1}^{\infty} n \left(\frac{3}{4}\right)^n = 1\left(\frac{3}{4}\right) + 2\left(\frac{9}{16}\right) + \dots \quad 6. \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \left(\frac{1}{n!}\right) = \frac{3}{4} + \frac{9}{16}\left(\frac{1}{2}\right) + \dots \quad 7. \sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n!} = 9 - \frac{3^3}{2} + \dots$$

$$S_1 = \frac{3}{4}, S_2 \approx 1.875$$

Matches (d).

$$S_1 = \frac{3}{4}, S_2 \approx 1.03$$

Matches (c).

$$S_1 = 9$$

Matches (f).

$$8. \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 4}{(2n)!} = \frac{4}{2} - \frac{4}{24} + \dots \quad 9. \sum_{n=1}^{\infty} \left(\frac{4n}{5n-3}\right)^n = \frac{4}{2} + \left(\frac{8}{7}\right)^2 + \dots \quad 10. \sum_{n=0}^{\infty} 4e^{-n} = 4 + \frac{4}{e} + \dots$$

$$S_1 = 2$$

Matches (b).

$$S_1 = 2, S_2 = 3.31$$

Matches (a).

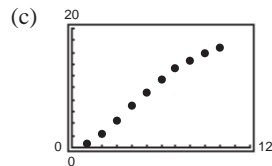
$$S_1 = 4$$

Matches (e).

$$11. (a) \text{ Ratio Test: } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^2 (5/8)^{n+1}}{n^2 (5/8)^n} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^2 \frac{5}{8} = \frac{5}{8} < 1. \text{ Converges}$$

(b)

|       |        |         |         |         |         |
|-------|--------|---------|---------|---------|---------|
| $n$   | 5      | 10      | 15      | 20      | 25      |
| $S_n$ | 9.2104 | 16.7598 | 18.8016 | 19.1878 | 19.2491 |



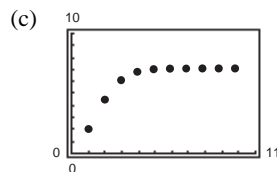
(d) The sum is approximately 19.26.

(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of the partial sums approaches the sum of the series.

$$12. (a) \text{ Ratio Test: } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^2 + 1}{n^2 + 1} = \lim_{n \rightarrow \infty} \left(\frac{n^2 + 2n + 2}{n^2 + 1}\right) \left(\frac{1}{n+1}\right) = 0 < 1. \text{ Converges}$$

(b)

|       |        |        |        |        |        |
|-------|--------|--------|--------|--------|--------|
| $n$   | 5      | 10     | 15     | 20     | 25     |
| $S_n$ | 7.0917 | 7.1548 | 7.1548 | 7.1548 | 7.1548 |



(d) The sum is approximately 7.15485

(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of the partial sums approaches the sum of the series.

$$13. \sum_{n=0}^{\infty} \frac{n!}{3^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{3} = \infty$$

Therefore, by the Ratio Test, the series diverges.

$$14. \sum_{n=0}^{\infty} \frac{3^n}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} \right| = \lim_{n \rightarrow \infty} \frac{3}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.



15.  $\sum_{n=1}^{\infty} n \left(\frac{3}{4}\right)^n$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)(3/4)^{n+1}}{n(3/4)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{3(n+1)}{4n} \right| = \frac{3}{4} \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

17.  $\sum_{n=1}^{\infty} \frac{n}{2^n}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{2n} = \frac{1}{2} \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

19.  $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{2n^2}{(n+1)^2} = 2 \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

21.  $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{2}{n+1} = 0 \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

23.  $\sum_{n=1}^{\infty} \frac{n!}{n3^n}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{(n+1)3^{n+1}} \cdot \frac{n3^n}{n!} \right| \\ &= \lim_{n \rightarrow \infty} \frac{n}{3} = \infty \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

16.  $\sum_{n=1}^{\infty} n \left(\frac{3}{2}\right)^n$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)3^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n3^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{3(n+1)}{2n} = \frac{3}{2} \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

18.  $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3/2^{n+1}}{n^3/2^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3}{2n^3} \right| = \frac{1}{2} \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

20.  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

$$\begin{aligned} a_{n+1} &= \frac{n+3}{(n+1)(n+2)} \leq \frac{n+2}{n(n+1)} = a_n \\ \lim_{n \rightarrow \infty} \frac{n+2}{n(n+1)} &= 0 \end{aligned}$$

Therefore, by Theorem 9.14, the series converges.

**Note:** The Ratio Test is inconclusive since  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ .

The series converges conditionally.

22.  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3/2)^n}{n^2}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(3/2)^{n+1}}{n^2 + 2n + 1} \cdot \frac{n^2}{(3/2)^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{3n^2}{2(n^2 + 2n + 1)} = \frac{3}{2} > 1 \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

24.  $\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(2n+2)!}{(n+1)^5} \cdot \frac{n^5}{(2n)!} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(2n+2)(2n+1)n^5}{(n+1)^5} = \infty \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

$$25. \sum_{n=0}^{\infty} \frac{4^n}{n!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{4^{n+1}}{(n+1)!} \cdot \frac{n!}{4^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{4}{n+1} = 0 \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

$$26. \sum_{n=1}^{\infty} \frac{n^n}{n!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)(n+1)^n n!}{(n+1)n!n^n} \\ &= \lim_{n \rightarrow \infty} \left( \frac{n+1}{n} \right)^n = e > 1 \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

$$27. \sum_{n=0}^{\infty} \frac{3^n}{(n+1)^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}}{(n+2)^{n+1}} \cdot \frac{(n+1)^n}{3^n} \right| = \lim_{n \rightarrow \infty} \frac{3(n+1)^n}{(n+2)^{n+1}} = \lim_{n \rightarrow \infty} \frac{3}{n+2} \left( \frac{n+1}{n+2} \right)^n = (0) \left( \frac{1}{e} \right) = 0$$

To find  $\lim_{n \rightarrow \infty} \left( \frac{n+1}{n+2} \right)^n$ , let  $y = \lim_{n \rightarrow \infty} \left( \frac{n+1}{n+2} \right)^n$ . Then,

$$\ln y = \lim_{n \rightarrow \infty} n \ln \left( \frac{n+1}{n+2} \right) = \lim_{n \rightarrow \infty} \frac{\ln[(n+1)/(n+2)]}{1/n} = \frac{0}{0}$$

$$\ln y = \lim_{n \rightarrow \infty} \frac{[(1)/(n+1)] - [(1)/(n+2)]}{-(1/n^2)} = -1 \text{ by L'Hôpital's Rule}$$

$$y = e^{-1} = \frac{1}{e}$$

Therefore, by the Ratio Test, the series converges.

$$28. \sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{[(n+1)!]^2 \cdot (3n)!}{(3n+3)! \cdot (n!)^2} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} = 0 \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

$$29. \sum_{n=0}^{\infty} \frac{4^n}{3^n + 1}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{4^{n+1}}{3^{n+1} + 1} \cdot \frac{3^n + 1}{4^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{4(3^n + 1)}{3^{n+1} + 1} \\ &= \lim_{n \rightarrow \infty} \frac{4(1 + 1/3^n)}{3 + 1/3^n} = \frac{4}{3} \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

$$30. \sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{4n+4}}{(2n+3)!} \cdot \frac{(2n+1)!}{2^{4n}} \right| = \lim_{n \rightarrow \infty} \frac{2^4}{(2n+3)(2n+2)} = 0$$

Therefore, by the Ratio Test, the series converges.

$$31. \sum_{n=0}^{\infty} \frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n+1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n+1)(2n+3)} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n+1)}{n!} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{2n+3} = \frac{1}{2}$$

Therefore, by the Ratio Test, the series converges.

**Note:** The first few terms of this series are  $-1 + \frac{1}{1 \cdot 3} - \frac{2!}{1 \cdot 3 \cdot 5} + \frac{3!}{1 \cdot 3 \cdot 5 \cdot 7} - \cdots$ .

$$32. \sum_{n=1}^{\infty} \frac{(-1)^n 2 \cdot 4 \cdot 6 \cdots 2n}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2 \cdot 4 \cdots 2n(2n+2)}{2 \cdot 5 \cdots (3n-1)(3n+2)} \cdot \frac{2 \cdot 5 \cdots (3n-1)}{2 \cdot 4 \cdots 2n} \right| = \lim_{n \rightarrow \infty} \frac{2n+2}{3n+2} = \frac{2}{3}$$

Therefore, by the Ratio Test, the series converges.

**Note:** The first few terms of this series are  $-\frac{2}{2} + \frac{2 \cdot 4}{2 \cdot 5} - \frac{2 \cdot 4 \cdot 6}{2 \cdot 5 \cdot 8} + \cdots$ .

$$33. \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{1} \right| \\ &= \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^{3/2} = 1 \end{aligned}$$

Ratio Test is inconclusive.

$$34. \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^{1/2}} \cdot \frac{n^{1/2}}{1} \right| \\ &= \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^{1/2} = 1 \end{aligned}$$

Ratio Test is inconclusive.

$$35. \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^4} \cdot \frac{n^4}{1} \right| = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^4 = 1$$

$$36. \sum_{n=1}^{\infty} \frac{1}{n^p}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^p} \cdot \frac{n^p}{1} \right| = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^p = 1$$

$$37. \sum_{n=1}^{\infty} \left( \frac{n}{2n+1} \right)^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{n}{2n+1} \right)^n} \\ &= \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2} \end{aligned}$$

Therefore, by the Root Test, the series converges.

$$38. \sum_{n=1}^{\infty} \left( \frac{2n}{n+1} \right)^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{2n}{n+1} \right)^n} \\ &= \lim_{n \rightarrow \infty} \frac{2n}{n+1} = 2 \end{aligned}$$

Therefore, by the Root Test, the series diverges.

$$39. \sum_{n=2}^{\infty} \left( \frac{2n+1}{n-1} \right)^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{2n+1}{n-1} \right)^n} \\ &= \lim_{n \rightarrow \infty} \left( \frac{2n+1}{n-1} \right) = 2 \end{aligned}$$

Since  $2 > 1$ , the series diverges.

$$40. \sum_{n=1}^{\infty} \left( \frac{4n+3}{2n-1} \right)^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{4n+3}{2n-1} \right)^n} \\ &= \lim_{n \rightarrow \infty} \frac{4n+3}{2n-1} = 2 \end{aligned}$$

Since  $2 > 1$ , the series diverges.

$$41. \sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{(-1)^n}{(\ln n)^n}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{|\ln n|} = 0 \end{aligned}$$

Therefore, by the Root Test, the series converges.

$$42. \sum_{n=1}^{\infty} \left( \frac{-3n}{2n+1} \right)^{3n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{-3n}{2n+1} \right)^{3n}} \\ &= \lim_{n \rightarrow \infty} \left( \frac{3n}{2n+1} \right)^3 = \left( \frac{3}{2} \right)^3 = \frac{27}{8} \end{aligned}$$

Therefore, by the Root Test, the series diverges.

$$43. \sum_{n=1}^{\infty} (2\sqrt[n]{n} + 1)^n$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{(2\sqrt[n]{n} + 1)^n} = \lim_{n \rightarrow \infty} (2\sqrt[n]{n} + 1)$$

To find  $\lim_{n \rightarrow \infty} \sqrt[n]{n}$ , let  $y = \lim_{n \rightarrow \infty} \sqrt[n]{n}$ . Then

$$\ln y = \lim_{n \rightarrow \infty} (\ln \sqrt[n]{n}) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln n = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{1/n}{1} = 0.$$

Thus,  $\ln y = 0$ , so  $y = e^0 = 1$  and  $\lim_{n \rightarrow \infty} (2\sqrt[n]{n} + 1) = 2(1) + 1 = 3$ . Therefore, by the Root Test, the series diverges.

$$44. \sum_{n=0}^{\infty} e^{-n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{e^n}} = \frac{1}{e}$$

Therefore, by the Root Test, the series converges.

$$45. \sum_{n=1}^{\infty} \frac{n}{4^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n}{4^n}} \\ &= \lim_{n \rightarrow \infty} \frac{n^{1/n}}{4} = \frac{1}{4} \end{aligned}$$

Since  $\frac{1}{4} < 1$ , the series converges.

**Note:** You can use L'Hôpital's Rule to show  $\lim_{n \rightarrow \infty} n^{1/n} = 1$ .

$$\text{Let } y = n^{1/n} \Rightarrow \ln y = \frac{\ln n}{n}$$

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{1/n}{1} = 0.$$

Hence,  $\ln y \rightarrow 0 \Rightarrow y = n^{1/n} \rightarrow 1$ .

$$46. \sum_{n=1}^{\infty} \left(\frac{n}{500}\right)^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{n}{500}\right)^n} \\ &= \lim_{n \rightarrow \infty} \left(\frac{n}{500}\right) = \infty \end{aligned}$$

The series diverges.

$$47. \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{1}{n} - \frac{1}{n^2}\right)^n} \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{n} - \frac{1}{n^2}\right) = 0 - 0 = 0 < 1 \end{aligned}$$

Hence, the series converges.

$$48. \sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{\ln n}{n}\right)^n} \\ &= \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0 < 1 \end{aligned}$$

By the Root Test, the series converges.

$$49. \sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n}{(\ln n)^n}} \\ &= \lim_{n \rightarrow \infty} \frac{n^{1/n}}{\ln n} = 0 \end{aligned}$$

Since  $0 < 1$ , the series converges by the Root Test.

$$50. \sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2} = \sum_{n=1}^{\infty} \frac{(n!)^n}{(n^2)^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{(n!)^n}{(n^2)^n}} \\ &= \lim_{n \rightarrow \infty} \frac{n!}{n^2} = \infty \end{aligned}$$

The series diverges.

$$51. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 5}{n}$$

$$a_{n+1} = \frac{5}{n+1} < \frac{5}{n} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{5}{n} = 0$$

Therefore, by the Alternating Series Test, the series converges (conditional convergence).

$$52. \sum_{n=1}^{\infty} \frac{5}{n} = 5 \sum_{n=1}^{\infty} \frac{1}{n}$$

This is the divergent harmonic series.

$$54. \sum_{n=1}^{\infty} \left(\frac{\pi}{4}\right)^n$$

Since  $\pi/4 < 1$ , this is convergent geometric series.

$$56. \sum_{n=1}^{\infty} \frac{n}{2n^2 + 1}$$

$$\lim_{n \rightarrow \infty} \frac{n/(2n^2 + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2}{2n^2 + 1} = \frac{1}{2} > 0$$

This series diverges by limit comparison to the divergent harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$58. \sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}}$$

$$\lim_{n \rightarrow \infty} \frac{10/3n^{3/2}}{1/n^{3/2}} = \frac{10}{3}$$

Therefore, the series converges by a Limit Comparison Test with the  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

$$60. \sum_{n=1}^{\infty} \frac{2^n}{4n^2 - 1}$$

$$\lim_{n \rightarrow \infty} \frac{2^n}{4n^2 - 1} = \lim_{n \rightarrow \infty} \frac{(\ln 2)2^n}{8n} = \lim_{n \rightarrow \infty} \frac{(\ln 2)^2 2^n}{8} = \infty$$

Therefore, the series diverges by the  $n$ th-Term Test for Divergence.

$$62. \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$$

$$a_{n+1} = \frac{1}{(n+1) \ln(n+1)} \leq \frac{1}{n \ln(n)} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n \ln(n)} = 0$$

Therefore, by the Alternating Series Test, the series converges.

$$53. \sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}} = 3 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

This is convergent  $p$ -series.

$$55. \sum_{n=1}^{\infty} \frac{2n}{n+1}$$

$$\lim_{n \rightarrow \infty} \frac{2n}{n+1} = 2 \neq 0$$

This diverges by the  $n$ th-Term Test for Divergence.

$$57. \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-2}}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^n 3^{-2}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{9} \left(-\frac{3}{2}\right)^n$$

Since  $|r| = \frac{3}{2} > 1$ , this is a divergent geometric series.

$$59. \sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$$

$$\lim_{n \rightarrow \infty} \frac{(10n+3)/n2^n}{1/2^n} = \lim_{n \rightarrow \infty} \frac{10n+3}{n} = 10$$

Therefore, the series converges by a Limit Comparison Test with the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$$61. \sum_{n=1}^{\infty} \frac{\cos n}{2^n}$$

$$\left| \frac{\cos n}{2^n} \right| \leq \frac{1}{2^n}$$

Therefore, the series

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{2^n} \right|$$

converges by comparison with the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$$63. \sum_{n=1}^{\infty} \frac{n7^n}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)7^{n+1}}{(n+1)!} \cdot \frac{n!}{n7^n} \right| = \lim_{n \rightarrow \infty} \frac{7}{n} = 0$$

Therefore, by the Ratio Test, the series converges.

$$64. \sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$$

$$\frac{\ln(n)}{n^2} \leq \frac{1}{n^{3/2}}$$

Therefore, the series converges by comparison with the  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

$$66. \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n2^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{3^n} \right| = \lim_{n \rightarrow \infty} \frac{3n}{2(n+1)} = \frac{3}{2}$$

Therefore, by the Ratio Test, the series diverges.

$$67. \sum_{n=1}^{\infty} \frac{(-3)^n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-3)^{n+1}}{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)} \cdot \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{(-3)^n} \right| = \lim_{n \rightarrow \infty} \frac{3}{2n+3} = 0$$

Therefore, by the Ratio Test, the series converges.

$$68. \sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{18^n (2n-1)n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)}{18^{n+1}(2n+1)(2n-1)n!} \cdot \frac{18^n (2n-1)n!}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \right| = \lim_{n \rightarrow \infty} \frac{(2n+3)(2n-1)}{18(2n+1)(2n-1)} = \frac{1}{18}$$

Therefore, by the Ratio Test, the series converge.

69. (a) and (c)

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n5^n}{n!} &= \sum_{n=0}^{\infty} \frac{(n+1)5^{n+1}}{(n+1)!} \\ &= 5 + \frac{(2)(5)^2}{2!} + \frac{(3)(5)^3}{3!} + \frac{(4)(5)^4}{4!} + \cdots \end{aligned}$$

71. (a) and (b) are the same.

$$65. \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-1}}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^n}{(n+1)!} \cdot \frac{n!}{3^{n-1}} \right| = \lim_{n \rightarrow \infty} \frac{3}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

(Absolutely)

70. (b) and (c)

$$\begin{aligned} \sum_{n=0}^{\infty} (n+1) \left(\frac{3}{4}\right)^n &= \sum_{n=1}^{\infty} n \left(\frac{3}{4}\right)^{n-1} \\ &= 1 + 2\left(\frac{3}{4}\right) + 3\left(\frac{3}{4}\right)^2 + 4\left(\frac{3}{4}\right)^3 + \cdots \end{aligned}$$

72. (a) and (b) are the same.

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)2^{n-1}} &= \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \cdots \\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n} &= \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \cdots \end{aligned}$$

73. Replace  $n$  with  $n+1$ .

$$\sum_{n=1}^{\infty} \frac{n}{4^n} = \sum_{n=0}^{\infty} \frac{n+1}{4^{n+1}}$$

74. Replace  $n$  with  $n+2$ .

$$\sum_{n=2}^{\infty} \frac{2^n}{(n-2)!} = \sum_{n=0}^{\infty} \frac{2^{n+2}}{n!}$$

75. Since

$$\frac{3^{10}}{2^{10} 10!} \approx 1.59 \times 10^{-5},$$

use 9 terms.

$$\sum_{k=1}^9 \frac{(-3)^k}{2^k k!} \approx -0.7769$$

$$\begin{aligned}
 76. \sum_{k=0}^{\infty} \frac{(-3)^k}{1 \cdot 3 \cdot 5 \cdots (2k+1)} &= \sum_{k=0}^{\infty} \frac{(-3)^k 2^k k!}{(2k)!(2k+1)} \\
 &= \sum_{k=0}^{\infty} \frac{(-6)^k k!}{(2k+1)!} \\
 &\approx 0.40967
 \end{aligned}$$

(See Exercise 3 and use 10 terms,  $k = 9$ .)

$$\begin{aligned}
 78. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(2n+1)/(5n-4)a_n}{a_n} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{2n+1}{5n-4} = \frac{2}{5} < 1
 \end{aligned}$$

The series converges by the Ratio Test.

$$\begin{aligned}
 80. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(\cos n + 1)/(n)a_n}{a_n} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{\cos n + 1}{n} = 0 < 1
 \end{aligned}$$

The series converges by the Ratio Test.

82. The series diverges because  $\lim_{n \rightarrow \infty} a_n \neq 0$ .

$$a_1 = \frac{1}{4}$$

$$a_2 = \left(\frac{1}{4}\right)^{1/2} = \frac{1}{2}$$

$$a_3 = \left(\frac{1}{2}\right)^{1/3} \approx 0.7937$$

In general,  $a_{n+1} > a_n > 0$ .

$$\begin{aligned}
 84. \sum_{n=0}^{\infty} \frac{n+1}{3^n} \\
 \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n+1}{3^n}} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n+1}}{3} \\
 \text{Let } y &= \lim_{n \rightarrow \infty} \sqrt[n]{n+1} \\
 \ln y &= \lim_{n \rightarrow \infty} (\ln \sqrt[n]{n+1}) \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \ln(n+1) \\
 &= \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{n} = \frac{1}{n+1} = 0.
 \end{aligned}$$

Since  $\ln y = 0$ ,  $y = e^0 = 1$ , so

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n]{n+1}}{3} = \frac{1}{3}.$$

Therefore, by the Root Test, the series converges.

$$\begin{aligned}
 77. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(4n-1)/(3n+2)a_n}{a_n} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{4n-1}{3n+2} = \frac{4}{3} > 1
 \end{aligned}$$

The series diverges by the Ratio Test.

$$\begin{aligned}
 79. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(\sin n + 1)/(\sqrt{n})a_n}{a_n} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{\sin n + 1}{\sqrt{n}} = 0 < 1
 \end{aligned}$$

The series converges by the Ratio Test.

$$\begin{aligned}
 81. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(1 + (1)/(n))a_n}{a_n} \right| \\
 &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 1
 \end{aligned}$$

The Ratio Test is inconclusive.

But,  $\lim_{n \rightarrow \infty} a_n \neq 0$ , so the series diverges.

$$\begin{aligned}
 83. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{1 \cdot 2 \cdots n(n+1)}{1 \cdot 3 \cdots (2n-1)(2n+1)}}{\frac{1 \cdot 2 \cdots n}{1 \cdot 3 \cdots (2n-1)}} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{n+1}{2n+1} = \frac{1}{2} < 1
 \end{aligned}$$

The series converges by the Ratio Test.

$$\begin{aligned}
 85. \sum_{n=3}^{\infty} \frac{1}{(\ln n)^n} \\
 \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0
 \end{aligned}$$

Therefore, by the Root Test, the series converges.

$$86. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{1 \cdot 2 \cdot 3 \cdots (2n-1)(2n)(2n+1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{1 \cdot 2 \cdot 3 \cdots (2n-1)} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{2n+1}{(2n)(2n+1)} = 0 < 1$$

The series converges by the Ratio Test.

$$88. \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{x+1}{4} \right|^n}$$

$$= \lim_{n \rightarrow \infty} \left| \frac{x+1}{4} \right| = \left| \frac{x+1}{4} \right|$$

For the series to converge,

$$\left| \frac{x+1}{4} \right| < 1 \Rightarrow -4 < x+1 < 4$$

$$\Rightarrow -5 < x < 3.$$

For  $x = 3$ ,  $\sum_{n=0}^{\infty} (1)^n$ , diverges.

For  $x = -5$ ,  $\sum_{n=0}^{\infty} (-1)^n$ , diverges.

Answer:  $-5 < x < 3$

$$90. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2|x-1|^{n+1}}{2|x-1|^n} \right|$$

$$= \lim_{n \rightarrow \infty} |x-1| = |x-1|$$

For the series to converge,

$$|x-1| < 1 \Rightarrow -1 < x-1 < 1$$

$$\Rightarrow 0 < x < 2.$$

For  $x = 2$ ,  $\sum_{n=0}^{\infty} 2(1)^n$ , diverges.

For  $x = 0$ ,  $\sum_{n=0}^{\infty} 2(-1)^n$ , diverges.

Answer:  $0 < x < 2$

$$92. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{|x+1|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x+1|^n} = \lim_{n \rightarrow \infty} \frac{|x+1|}{n+1} = 0$$

The series converges for all  $x$ .

93. See Theorem 9.17, page 597.

$$87. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2(x/3)^{n+1}}{2(x/3)^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{x}{3} \right| = \left| \frac{x}{3} \right|$$

For the series to converge:  $\left| \frac{x}{3} \right| < 1 \Rightarrow -3 < x < 3$ .

For  $x = 3$ , the series diverges.

For  $x = -3$ , the series diverges.

Answer:  $-3 < x < 3$

$$89. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x+1)^{n+1}/(n+1)}{x^n/n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{n}{n+1}(x+1) \right| = |x+1|$$

For the series to converge,

$$|x+1| < 1 \Rightarrow -1 < x+1 < 1$$

$$\Rightarrow -2 < x < 0.$$

For  $x = 0$ ,  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ , converges.

For  $x = -2$ ,  $\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ , diverges.

Answer:  $-2 < x \leq 0$

$$91. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)! \left| \frac{x}{2} \right|^{n+1}}{n! \left| \frac{x}{2} \right|^n}$$

$$= \lim_{n \rightarrow \infty} (n+1) \left| \frac{x}{2} \right| = \infty$$

The series converges only at  $x = 0$ .

94. See Theorem 9.18.



95. No. Let  $a_n = \frac{1}{n + 10,000}$ .

The series  $\sum_{n=1}^{\infty} \frac{1}{n + 10,000}$  diverges.

97. The series converges absolutely. See Theorem 9.17.

98. Assume that

$$\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = L > 1 \text{ or that } \lim_{n \rightarrow \infty} |a_{n+1}/a_n| = \infty.$$

Then there exists  $N > 0$  such that  $|a_{n+1}/a_n| > 1$  for all  $n > N$ . Therefore,

$$|a_{n+1}| > |a_n|, \quad n > N \Rightarrow \lim_{n \rightarrow \infty} a_n \neq 0 \Rightarrow \sum a_n \text{ diverges.}$$

99. First, let

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = r < 1$$

and choose  $R$  such that  $0 \leq r < R < 1$ . There must exist some  $N > 0$  such that  $\sqrt[n]{|a_n|} < R$  for all  $n > N$ . Thus, for  $n > N$   $|a_n| < R^n$  and since the geometric series

$$\sum_{n=0}^{\infty} R^n$$

converges, we can apply the Comparison Test to conclude that

$$\sum_{n=1}^{\infty} |a_n|$$

converges which in turn implies that  $\sum_{n=1}^{\infty} a_n$  converges.

100.  $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ,  $p$ -series

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n^p}} = \lim_{n \rightarrow \infty} \frac{1}{n^{p/n}} = 1$$

Hence, the Root Test is inconclusive.

**Note:**  $\lim_{n \rightarrow \infty} n^{p/n} = 1$  because if  $y = n^{p/n}$ , then

$$\ln y = \frac{p}{n} \ln n \text{ and } \frac{p}{n} \ln n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence  $y \rightarrow 1$  as  $n \rightarrow \infty$ .

96. One example is  $\sum_{n=1}^{\infty} \left(-100 + \frac{1}{n}\right)$ .

Second, let

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = r > R > 1.$$

Then there must exist some  $M > 0$  such that  $\sqrt[n]{|a_n|} > R$  for infinitely many  $n > M$ . Thus, for infinitely many  $n > M$ , we have  $|a_n| > R^n > 1$  which implies that  $\lim_{n \rightarrow \infty} a_n \neq 0$  which in turn implies that

$$\sum_{n=1}^{\infty} a_n \text{ diverges.}$$

101. Ratio Test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n(\ln n)^p}{(n+1)(\ln(n+1))^p} = 1, \text{ inconclusive.}$$

Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n(\ln n)^p}} = \lim_{n \rightarrow \infty} \frac{1}{n^{1/n}(\ln n)^{p/n}}$$

$$\lim_{n \rightarrow \infty} n^{1/n} = 1. \text{ Furthermore, let } y = (\ln n)^{p/n} \Rightarrow$$

$$\ln y = \frac{p}{n} \ln(\ln n). \quad \lim_{n \rightarrow \infty} \ln y = \lim_{n \rightarrow \infty} \frac{p \ln(\ln n)}{n}$$

$$= \lim_{n \rightarrow \infty} \frac{p}{\ln(n)(1/n)} = 0 \Rightarrow \lim_{n \rightarrow \infty} (\ln n)^{p/n} = 1.$$

$$\text{Thus, } \lim_{n \rightarrow \infty} \frac{1}{n^{1/n}(\ln n)^{p/n}} = 1, \text{ inconclusive.}$$

102.  $\sum_{n=1}^{\infty} \frac{(n!)^2}{(xn)!}$ ,  $x$  positive integer

(a)  $x = 1$ :  $\sum \frac{(n!)^2}{n!} = \sum n!$ , diverges

(b)  $x = 2$ :  $\sum \frac{(n!)^2}{(2n)!}$  converges by the Ratio Test:

$$\lim_{n \rightarrow \infty} \frac{[(n+1)!]^2}{(2n+2)!} \bigg/ \frac{(n!)^2}{(2n)!} = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$$

(c)  $x = 3$ :  $\sum \frac{(n!)^2}{(3n)!}$  converges by the Ratio Test:

$$\lim_{n \rightarrow \infty} \frac{[(n+1)!]^2}{(3n+3)!} \bigg/ \frac{(n!)^2}{(3n)!} = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} = 0 < 1$$

(d) Use the Ratio Test:

$$\lim_{n \rightarrow \infty} \frac{[(n+1)!]^2}{[x(n+1)]!} \bigg/ \frac{(n!)^2}{(xn)!} = \lim_{n \rightarrow \infty} (n+1)^2 \frac{(xn)!}{(xn+x)!}$$

The cases  $x = 1, 2, 3$  were solved above. For  $x > 3$ , the limit is 0. Hence, the series converges for all integers  $x \geq 2$ .

103. For  $n = 1, 2, 3, \dots$ ,  $-|a_n| \leq a_n \leq |a_n| \Rightarrow -\sum_{n=1}^k |a_n| \leq \sum_{n=1}^k a_n \leq \sum_{n=1}^k |a_n|$ .

Taking limits as  $k \rightarrow \infty$ ,  $-\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} |a_n| \Rightarrow \left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|$ .

104. The differentiation test states that if

$$\sum_{n=1}^{\infty} U_n$$

is an infinite series with real terms and  $f(x)$  is a real function such that  $f(1/n) = U_n$  for all positive integers  $n$  and  $d^2 f/dx^2$  exists at  $x = 0$ , then

$$\sum_{n=1}^{\infty} U_n$$

converges absolutely if  $f(0) = f'(0) = 0$  and diverges otherwise. Below are some examples.

Convergent Series

$$\sum \frac{1}{n^3}, f(x) = x^3$$

$$\sum \left(1 - \cos \frac{1}{n}\right), f(x) = 1 - \cos x$$

Divergent Series

$$\sum \frac{1}{n}, f(x) = x$$

$$\sum \sin \frac{1}{n}, f(x) = \sin x$$

105. Using the Ratio Test,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left[ \frac{n!}{(n+1)^n} \left(\frac{19}{7}\right)^n \bigg/ \frac{(n-1)!}{n^{n-1}} \left(\frac{19}{7}\right)^{n-1} \right] \\ &= \lim_{n \rightarrow \infty} \left[ \frac{n \cdot n^{n-1}}{(n+1)^n} \left(\frac{19}{7}\right) \right] \\ &= \lim_{n \rightarrow \infty} \left[ \frac{1}{\left(1 + \frac{1}{n}\right)^n} \left(\frac{19}{7}\right) \right] \\ &= \frac{19}{7} \cdot \frac{1}{e} < 1 \end{aligned}$$

Hence, the series converges.

106. We first prove Abel's Summation Theorem:

If the partial sums of  $\sum a_n$  are bounded and if  $\{b_n\}$  decreases to zero, then  $\sum a_n b_n$  converges.

Let  $S_k = \sum_{i=1}^k a_i$ . Let  $M$  be a bound for  $\{|S_k|\}$ .

$$\begin{aligned} a_1 b_1 + a_2 b_2 + \cdots + a_n b_n &= S_1 b_1 + (S_2 - S_1) b_2 + \cdots + (S_n - S_{n-1}) b_n \\ &= S_1(b_1 - b_2) + S_2(b_2 - b_3) + \cdots + S_{n-1}(b_{n-1} - b_n) + S_n b_n \\ &= \sum_{i=1}^{n-1} S_i(b_i - b_{i+1}) + S_n b_n \end{aligned}$$

The series  $\sum_{i=1}^{\infty} S_i(b_i - b_{i+1})$  is absolutely convergent because  $|S_i(b_i - b_{i+1})| \leq M(b_i - b_{i+1})$  and

$\sum_{i=1}^{\infty} (b_i - b_{i+1})$  converges to  $b_1$ .

Also,  $\lim_{n \rightarrow \infty} S_n b_n = 0$  because  $\{S_n\}$  bounded and  $b_n \rightarrow 0$ . Thus,  $\sum_{n=1}^{\infty} a_n b_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i b_i$  converges.

Now let  $b_n = \frac{1}{n}$  to finish the problem.

## Section 9.7 Taylor Polynomials and Approximations

1.  $y = -\frac{1}{2}x^2 + 1$

Parabola

Matches (d)

2.  $y = \frac{1}{8}x^4 - \frac{1}{2}x^2 + 1$

y-axis symmetry

Three relative extrema

Matches (c)

3.  $y = e^{-1/2}[(x+1) + 1]$

Linear

Matches (a)

4.  $y = e^{-1/2}[\frac{1}{3}(x-1)^3 - (x-1) + 1]$

Cubic

Matches (b)

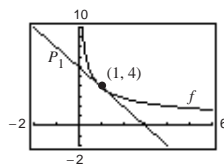
5.  $f(x) = \frac{4}{\sqrt{x}} = 4x^{-1/2} \quad f(1) = 4$

$f'(x) = -2x^{-3/2} \quad f'(1) = -2$

$P_1(x) = f(1) + f'(1)(x-1)$

$= 4 + (-2)(x-1)$

$P_1(x) = -2x + 6$



$P_1$  is called the first degree Taylor polynomial for  $f$  at  $c$ .

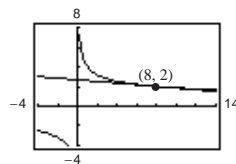
6.  $f(x) = \frac{4}{\sqrt[3]{x}} = 4x^{-1/3} \quad f(8) = 2$

$f'(x) = -\frac{4}{3}x^{-4/3} \quad f'(8) = -\frac{1}{12}$

$P_1(x) = f(8) + f'(8)(x-8)$

$= 2 + \left(-\frac{1}{12}\right)(x-8)$

$P_1(x) = -\frac{1}{12}x + \frac{8}{3}$



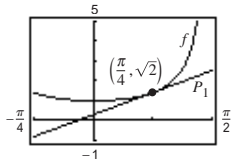
$P_1$  is called the first degree Taylor polynomial for  $f$  at  $c$ .

$$7. f(x) = \sec x \quad f\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$f'(x) = \sec x \tan x \quad f'\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$P_1(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$$

$$P_1(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right)$$



$P_1$  is called the first degree Taylor polynomial for  $f$  at  $c$ .

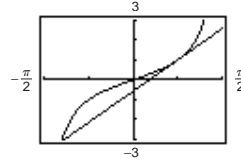
$$8. f(x) = \tan x \quad f\left(\frac{\pi}{4}\right) = 1$$

$$f'(x) = \sec^2 x \quad f'\left(\frac{\pi}{4}\right) = 2$$

$$P_1 = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$$

$$= 1 + 2\left(x - \frac{\pi}{4}\right)$$

$$P_1(x) = 2x + 1 - \frac{\pi}{2}$$



$P_1$  is called the first degree Taylor polynomial for  $f$  at  $c$ .

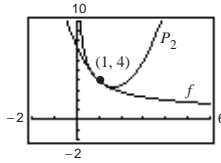
$$9. f(x) = \frac{4}{\sqrt{x}} = 4x^{-1/2} \quad f(1) = 4$$

$$f'(x) = -2x^{-3/2} \quad f'(1) = -2$$

$$f''(x) = 3x^{-5/2} \quad f''(1) = 3$$

$$P_2 = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2}(x - 1)^2$$

$$= 4 - 2(x - 1) + \frac{3}{2}(x - 1)^2$$



| $x$      | 0     | 0.8    | 0.9    | 1.0 | 1.1    | 1.2    | 2      |
|----------|-------|--------|--------|-----|--------|--------|--------|
| $f(x)$   | Error | 4.4721 | 4.2164 | 4.0 | 3.8139 | 3.6515 | 2.8284 |
| $P_2(x)$ | 7.5   | 4.46   | 4.215  | 4.0 | 3.815  | 3.66   | 3.5    |

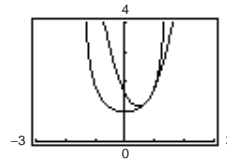
$$10. f(x) = \sec x \quad f\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$f'(x) = \sec x \tan x \quad f'\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$f''(x) = \sec^3 x + \sec x \tan^2 x \quad f''\left(\frac{\pi}{4}\right) = 3\sqrt{2}$$

$$P_2(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''(\pi/4)}{2}\left(x - \frac{\pi}{4}\right)^2$$

$$P_2(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3}{2}\sqrt{2}\left(x - \frac{\pi}{4}\right)^2$$



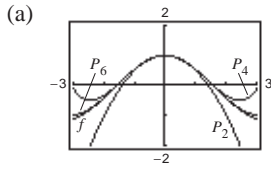
| $x$      | -2.15   | 0.585  | 0.685  | $\pi/4$ | 0.885  | 0.985  | 1.785   |
|----------|---------|--------|--------|---------|--------|--------|---------|
| $f(x)$   | -1.8270 | 1.1995 | 1.2913 | 1.4142  | 1.5791 | 1.8088 | -4.7043 |
| $P_2(x)$ | 15.5414 | 1.2160 | 1.2936 | 1.4142  | 1.5761 | 1.7810 | 4.9475  |

11.  $f(x) = \cos x$

$$P_2(x) = 1 - \frac{1}{2}x^2$$

$$P_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

$$P_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$$



(b)  $f'(x) = -\sin x$   $P_2'(x) = -x$

$$f''(x) = -\cos x$$
  $P_2''(x) = -1$

$$f''(0) = P_2''(0) = -1$$

$$f'''(x) = \sin x$$
  $P_4'''(x) = x$

$$f^{(4)}(x) = \cos x$$
  $P_4^{(4)}(x) = 1$

$$f^{(4)}(0) = 1 = P_4^{(4)}(0)$$

$$f^{(5)}(x) = -\sin x$$
  $P_6^{(5)}(x) = -x$

$$f^{(6)}(x) = -\cos x$$
  $P_6^{(6)}(x) = -1$

$$f^{(6)}(0) = -1 = P_6^{(6)}(0)$$

(c) In general,  $f^{(n)}(0) = P_n^{(n)}(0)$  for all  $n$ .

12.  $f(x) = x^2e^x, f(0) = 0$

(a)  $f'(x) = (x^2 + 2x)e^x$   $f'(0) = 0$

$$f''(x) = (x^2 + 4x + 2)e^x$$
  $f''(0) = 2$

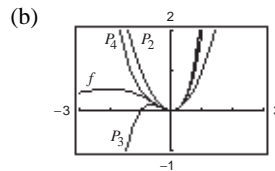
$$f'''(x) = (x^2 + 6x + 6)e^x$$
  $f'''(0) = 6$

$$f^{(4)}(x) = (x^2 + 8x + 12)e^x$$
  $f^{(4)}(0) = 12$

$$P_2(x) = \frac{2x^2}{2!} = x^2$$

$$P_3(x) = x^2 + \frac{6x^3}{3!} = x^2 + x^3$$

$$P_4(x) = x^2 + x^3 + \frac{12x^4}{4!} = x^2 + x^3 + \frac{x^4}{2}$$



(c)  $f''(0) = 2 = P_2''(0)$

$$f'''(0) = 6 = P_3'''(0)$$

$$f^{(4)}(0) = 12 = P_4^{(4)}(0)$$

(d)  $f^{(n)}(0) = P_n^{(n)}(0)$

13.  $f(x) = e^{-x}$   $f(0) = 1$

$$f'(x) = -e^{-x}$$
  $f'(0) = -1$

$$f''(x) = e^{-x}$$
  $f''(0) = 1$

$$f'''(x) = -e^{-x}$$
  $f'''(0) = -1$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$$

14.  $f(x) = e^{-x}$   $f(0) = 1$

$$f'(x) = -e^{-x}$$
  $f'(0) = -1$

$$f''(x) = e^{-x}$$
  $f''(0) = 1$

$$f'''(x) = -e^{-x}$$
  $f'''(0) = -1$

$$f^{(4)}(x) = e^{-x}$$
  $f^{(4)}(0) = 1$

$$f^{(5)}(x) = -e^{-x}$$
  $f^{(5)}(0) = -1$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$+ \frac{f^{(5)}(0)}{5!}x^5 = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$$

15.  $f(x) = e^{2x}$   $f(0) = 1$

$$f'(x) = 2e^{2x}$$
  $f'(0) = 2$

$$f''(x) = 4e^{2x}$$
  $f''(0) = 4$

$$f'''(x) = 8e^{2x}$$
  $f'''(0) = 8$

$$f^{(4)}(x) = 16e^{2x}$$
  $f^{(4)}(0) = 16$

$$P_4(x) = 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4$$

16.  $f(x) = e^{3x}$   $f(0) = 1$

$$f'(x) = 3e^{3x}$$
  $f'(0) = 3$

$$f''(x) = 9e^{3x}$$
  $f''(0) = 9$

$$f'''(x) = 27e^{3x}$$
  $f'''(0) = 27$

$$f^{(4)}(x) = 81e^{3x}$$
  $f^{(4)}(0) = 81$

$$P_4(x) = 1 + 3x + \frac{9}{2!}x^2 + \frac{27}{3!}x^3 + \frac{81}{4!}x^4$$

$$= 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4$$

$$\begin{aligned}
 17. \quad f(x) &= \sin x & f(0) &= 0 \\
 f'(x) &= \cos x & f'(0) &= 1 \\
 f''(x) &= -\sin x & f''(0) &= 0 \\
 f'''(x) &= -\cos x & f'''(0) &= -1 \\
 f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\
 f^{(5)}(x) &= \cos x & f^{(5)}(0) &= 1
 \end{aligned}$$

$$\begin{aligned}
 P_5(x) &= 0 + (1)x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 \\
 &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5
 \end{aligned}$$

$$\begin{aligned}
 19. \quad f(x) &= xe^x & f(0) &= 0 \\
 f'(x) &= xe^x + e^x & f'(0) &= 1 \\
 f''(x) &= xe^x + 2e^x & f''(0) &= 2 \\
 f'''(x) &= xe^x + 3e^x & f'''(0) &= 3 \\
 f^{(4)}(x) &= xe^x + 4e^x & f^{(4)}(0) &= 4
 \end{aligned}$$

$$\begin{aligned}
 P_4(x) &= 0 + x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{4}{4!}x^4 \\
 &= x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4
 \end{aligned}$$

$$\begin{aligned}
 21. \quad f(x) &= \frac{1}{x+1} & f(0) &= 1 \\
 f'(x) &= -\frac{1}{(x+1)^2} & f'(0) &= -1 \\
 f''(x) &= \frac{2}{(x+1)^3} & f''(0) &= 2 \\
 f'''(x) &= \frac{-6}{(x+1)^4} & f'''(0) &= -6 \\
 f^{(4)}(x) &= \frac{24}{(x+1)^5} & f^{(4)}(0) &= 24
 \end{aligned}$$

$$\begin{aligned}
 P_4(x) &= 1 - x + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{24}{4!}x^4 \\
 &= 1 - x + x^2 - x^3 + x^4
 \end{aligned}$$

$$\begin{aligned}
 23. \quad f(x) &= \sec x & f(0) &= 1 \\
 f'(x) &= \sec x \tan x & f'(0) &= 0 \\
 f''(x) &= \sec^3 x + \sec x \tan^2 x & f''(0) &= 1
 \end{aligned}$$

$$P_2(x) = 1 + 0x + \frac{1}{2!}x^2 = 1 + \frac{1}{2}x^2$$

$$\begin{aligned}
 18. \quad f(x) &= \sin \pi x & f(0) &= 0 \\
 f'(x) &= \pi \cos \pi x & f'(0) &= \pi \\
 f''(x) &= -\pi^2 \sin \pi x & f''(0) &= 0 \\
 f'''(x) &= -\pi^3 \cos \pi x & f'''(0) &= -\pi^3
 \end{aligned}$$

$$P_3(x) = 0 + \pi x + \frac{0}{2!}x^2 + \frac{-\pi^3}{3!}x^3 = \pi x - \frac{\pi^3}{6}x^3$$

$$\begin{aligned}
 20. \quad f(x) &= x^2e^{-x} & f(0) &= 0 \\
 f'(x) &= 2xe^{-x} - x^2e^{-x} & f'(0) &= 0 \\
 f''(x) &= 2e^{-x} - 4xe^{-x} + x^2e^{-x} & f''(0) &= 2 \\
 f'''(x) &= -6e^{-x} + 6xe^{-x} - x^2e^{-x} & f'''(0) &= -6 \\
 f^{(4)}(x) &= 12e^{-x} - 8xe^{-x} + x^2e^{-x} & f^{(4)}(0) &= 12
 \end{aligned}$$

$$\begin{aligned}
 P_4(x) &= 0 + 0x + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{12}{4!}x^4 \\
 &= x^2 - x^3 + \frac{1}{2}x^4
 \end{aligned}$$

$$\begin{aligned}
 22. \quad f(x) &= \frac{x}{x+1} = \frac{x+1-1}{x+1} & f(0) &= 0 \\
 &= 1 - (x+1)^{-1} \\
 f'(x) &= (x+1)^{-2} & f'(0) &= 1 \\
 f''(x) &= -2(x+1)^{-3} & f''(0) &= -2 \\
 f'''(x) &= 6(x+1)^{-4} & f'''(0) &= 6 \\
 f^{(4)}(x) &= -24(x+1)^{-5} & f^{(4)}(0) &= -24 \\
 P_4(x) &= 0 + 1(x) - \frac{2}{2}x^2 + \frac{6}{6}x^3 - \frac{24}{24}x^4 \\
 &= x - x^2 + x^3 - x^4
 \end{aligned}$$

$$\begin{aligned}
 24. \quad f(x) &= \tan x & f(0) &= 0 \\
 f'(x) &= \sec^2 x & f'(0) &= 1 \\
 f''(x) &= 2 \sec^2 x \tan x & f''(0) &= 0 \\
 f'''(x) &= 4 \sec^2 x \tan^2 x + 2 \sec^4 x & f'''(0) &= 2
 \end{aligned}$$

$$P_3(x) = 0 + 1(x) + 0 + \frac{2}{6}x^3 = x + \frac{1}{3}x^3$$

$$25. \quad f(x) = \frac{1}{x} \quad f(1) = 1$$

$$f'(x) = -\frac{1}{x^2} \quad f'(1) = -1$$

$$f''(x) = \frac{2}{x^3} \quad f''(1) = 2$$

$$f'''(x) = -\frac{6}{x^4} \quad f'''(1) = -6$$

$$f^{(4)}(x) = \frac{24}{x^5} \quad f^{(4)}(1) = 24$$

$$P_4(x) = 1 - (x-1) + \frac{2}{2!}(x-1)^2 + \frac{-6}{3!}(x-1)^3$$

$$+ \frac{24}{4!}(x-1)^4$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4$$

$$27. \quad f(x) = \sqrt{x} \quad f(1) = 1$$

$$f'(x) = \frac{1}{2\sqrt{x}} \quad f'(1) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4x\sqrt{x}} \quad f''(1) = -\frac{1}{4}$$

$$f'''(x) = \frac{3}{8x^2\sqrt{x}} \quad f'''(1) = \frac{3}{8}$$

$$f^{(4)}(x) = -\frac{15}{16x^3\sqrt{x}} \quad f^{(4)}(1) = -\frac{15}{16}$$

$$P_4(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$$

$$+ \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4$$

$$29. \quad f(x) = \ln x \quad f(1) = 0$$

$$f'(x) = \frac{1}{x} \quad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \quad f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \quad f'''(1) = 2$$

$$f^{(4)}(x) = -\frac{6}{x^4} \quad f^{(4)}(1) = -6$$

$$P_4(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2$$

$$+ \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

$$26. \quad f(x) = 2x^{-2} \quad f(2) = \frac{1}{2}$$

$$f'(x) = -4x^{-3} \quad f'(2) = -\frac{1}{2}$$

$$f''(x) = 12x^{-4} \quad f''(2) = \frac{3}{4}$$

$$f'''(x) = -48x^{-5} \quad f'''(2) = -\frac{3}{2}$$

$$f^{(4)}(x) = 240x^{-6} \quad f^{(4)}(2) = \frac{15}{4}$$

$$P_4(x) = \frac{1}{2} - \frac{1}{2}(x-2) + \frac{3}{8}(x-2)^2 - \frac{1}{4}(x-2)^3$$

$$+ \frac{5}{32}(x-2)^4$$

$$28. \quad f(x) = x^{1/3} \quad f(8) = 2$$

$$f'(x) = \frac{1}{3}x^{-2/3} \quad f'(8) = \frac{1}{12}$$

$$f''(x) = -\frac{2}{9}x^{-5/3} \quad f''(8) = -\frac{1}{144}$$

$$f'''(x) = \frac{10}{27}x^{-8/3} \quad f'''(8) = \frac{10}{27} \cdot \frac{1}{2^8} = \frac{5}{3456}$$

$$P_3(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20,736}(x-8)^3$$

$$30. \quad f(x) = x^2 \cos x \quad f(\pi) = -\pi^2$$

$$f'(x) = \cos x - x^2 \sin x \quad f'(\pi) = -2\pi$$

$$f''(x) = 2 \cos x - 4x \sin x - x^2 \cos x \quad f''(\pi) = -2 + \pi^2$$

$$P_2(x) = -\pi^2 - 2\pi(x-\pi) + \frac{(\pi^2-2)}{2}(x-\pi)^2$$

31.  $f(x) = \tan x$

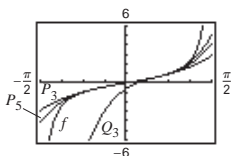
$f'(x) = \sec^2 x$

$f''(x) = 2 \sec^2 x \tan x$

$f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x$

$f^{(4)}(x) = 8 \sec^2 x \tan^3 x + 16 \sec^4 x \tan x$

$f^{(5)}(x) = 16 \sec^2 x \tan^4 x + 88 \sec^4 x \tan^2 x + 16 \sec^6 x$



(a)  $n = 3, c = 0$

$$P_3(x) = 0 + x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 = x + \frac{1}{3}x^3$$

(b)  $n = 3, c = \frac{\pi}{4}$

$$\begin{aligned} Q_3(x) &= 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{16}{3!}\left(x - \frac{\pi}{4}\right)^3 \\ &= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 \end{aligned}$$

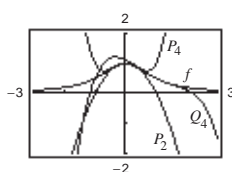
32.  $f(x) = \frac{1}{x^2 + 1}$

$f'(x) = \frac{-2x}{(x^2 + 1)^2}$

$f''(x) = \frac{2(3x^2 - 1)}{(x^2 + 1)^3}$

$f'''(x) = \frac{24x(1 - x^2)}{(x^2 + 1)^4}$

$f^{(4)}(x) = \frac{24(5x^4 - 10x^2 + 1)}{(x^2 + 1)^5}$



(a)  $n = 4, c = 0$

$$P_4(x) = 1 + 0x + \frac{-2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{24}{4!}x^4 = 1 - x^2 + x^4$$

(b)  $n = 4, c = 1$

$$Q_4(x) = \frac{1}{2} + \left(-\frac{1}{2}\right)(x - 1) + \frac{1/2}{2!}(x - 1)^2 + \frac{0}{3!}(x - 1)^3 + \frac{-3}{4!}(x - 1)^4 = \frac{1}{2} - \frac{1}{2}(x - 1) + \frac{1}{4}(x - 1)^2 - \frac{1}{8}(x - 1)^4$$

33.  $f(x) = \sin x$

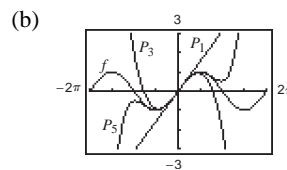
$P_1(x) = x$

$P_3(x) = x - \frac{1}{6}x^3$

$P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

(a)

| $x$      | 0.00   | 0.25   | 0.50   | 0.75   | 1.00   |
|----------|--------|--------|--------|--------|--------|
| $\sin x$ | 0.0000 | 0.2474 | 0.4794 | 0.6816 | 0.8415 |
| $P_1(x)$ | 0.0000 | 0.2500 | 0.5000 | 0.7500 | 1.0000 |
| $P_3(x)$ | 0.0000 | 0.2474 | 0.4792 | 0.6797 | 0.8333 |
| $P_5(x)$ | 0.0000 | 0.2474 | 0.4794 | 0.6817 | 0.8417 |



(c) As the distance increases, the accuracy decreases.



34.  $f(x) = \ln x, c = 1$

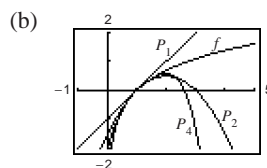
$P_1(x) = x - 1$

$P_2(x) = (x - 1) - \frac{1}{2}(x - 1)^2$

$P_4(x) = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4$

(a)

|          |      |        |        |        |        |
|----------|------|--------|--------|--------|--------|
| $x$      | 1.00 | 1.25   | 1.50   | 1.75   | 2.00   |
| $\ln x$  | 0    | 0.2231 | 0.4055 | 0.5596 | 0.6931 |
| $P_1(x)$ | 0    | 0.2500 | 0.5000 | 0.7500 | 1.0000 |
| $P_2(x)$ | 0    | 0.2188 | 0.375  | 0.4688 | 0.5000 |
| $P_4(x)$ | 0    | 0.2230 | 0.4010 | 0.5303 | 0.5833 |



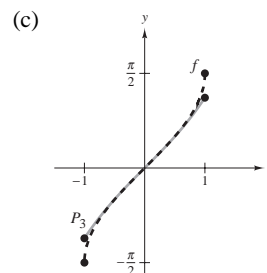
(c) As the degree increases, the accuracy increases. As the distance from  $x$  to 1 increases, the accuracy decreases.

35.  $f(x) = \arcsin x$

(a)  $P_3(x) = x + \frac{x^3}{6}$

(b)

|          |        |        |        |   |       |       |       |
|----------|--------|--------|--------|---|-------|-------|-------|
| $x$      | -0.75  | -0.50  | -0.25  | 0 | 0.25  | 0.50  | 0.75  |
| $f(x)$   | -0.848 | -0.524 | -0.253 | 0 | 0.253 | 0.524 | 0.848 |
| $P_3(x)$ | -0.820 | -0.521 | -0.253 | 0 | 0.253 | 0.521 | 0.820 |

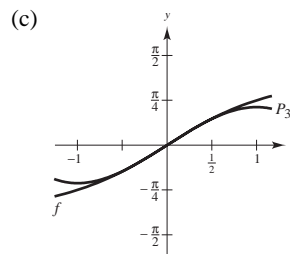


36. (a)  $f(x) = \arctan x$

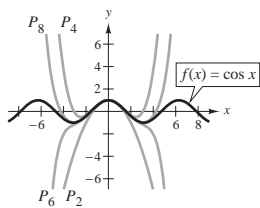
$P_3(x) = x - \frac{x^3}{3}$

(b)

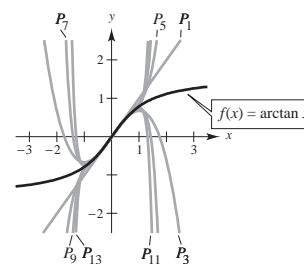
|          |         |         |         |   |        |        |        |
|----------|---------|---------|---------|---|--------|--------|--------|
| $x$      | -0.75   | -0.50   | -0.25   | 0 | 0.25   | 0.50   | 0.75   |
| $f(x)$   | -0.6435 | -0.4636 | -0.2450 | 0 | 0.2450 | 0.4636 | 0.6435 |
| $P_3(x)$ | -0.6094 | -0.4583 | -0.2448 | 0 | 0.2448 | 0.4583 | 0.6094 |



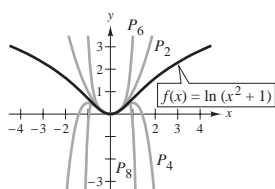
37.  $f(x) = \cos x$



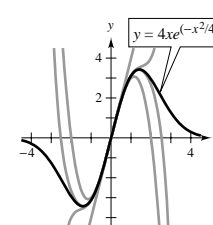
38.  $f(x) = \arctan x$



39.  $f(x) = \ln(x^2 + 1)$



40.  $f(x) = 4xe^{-x^2/4}$



$$41. f(x) = e^{-x} \approx 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$$

$$f\left(\frac{1}{2}\right) \approx 0.6042$$

$$42. f(x) = x^2 e^{-x} \approx x^2 - x^3 + \frac{1}{2}x^4$$

$$f\left(\frac{1}{5}\right) \approx 0.0328$$

$$43. f(x) = \ln x \approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

$$f(1.2) \approx 0.1823$$

$$44. f(x) = x^2 \cos x \approx -\pi^2 - 2\pi(x-\pi) + \left(\frac{\pi^2-2}{2}\right)(x-\pi)^2$$

$$f\left(\frac{7\pi}{8}\right) \approx -6.7954$$

$$45. f(x) = \cos x; f^{(5)}(x) = -\sin x \Rightarrow \text{Max on } [0, 0.3] \text{ is } 1.$$

$$R_4(x) \leq \frac{1}{5!}(0.3)^5 = 2.025 \times 10^{-5}$$

Note: you could use  $R_5(x)$ :  $f^{(6)}(x) = -\cos x$ , max on  $[0, 0.3]$  is 1.

$$R_5(x) \leq \frac{1}{6!}(0.3)^6 = 1.0125 \times 10^{-6}$$

Exact error:  $0.000001 = 1.0 \times 10^{-6}$

$$46. f(x) = e^x; f^{(6)}(x) = e^x \Rightarrow \text{Max on } [0, 1] \text{ is } e^1.$$

$$R_5(x) \leq \frac{e^1}{6!}(1)^6 \approx 0.00378 = 3.78 \times 10^{-3}$$

$$47. f(x) = \arcsin x; f^{(4)}(x) = \frac{x(6x^2+9)}{(1-x^2)^{7/2}} \Rightarrow \text{Max on } [0, 0.4] \text{ is } f^{(4)}(0.4) \approx 7.3340.$$

$$R_3(x) \leq \frac{7.3340}{4!}(0.4)^4 \approx 0.00782 = 7.82 \times 10^{-3}. \text{ The exact error is } 8.5 \times 10^{-4}. \text{ [Note: You could use } R_4.]$$

$$48. f(x) = \arctan x; f^{(4)}(x) = \frac{24x(x^2+1)}{(1-x^2)^4}$$

$$\Rightarrow \text{Max on } [0, 0.4] \text{ is } f^{(4)}(0.4) \approx 22.3672.$$

$$R_3(x) \leq \frac{22.3672}{4!}(0.4)^4 \approx 0.0239$$

$$49. g(x) = \sin x$$

$$|g^{(n+1)}(x)| \leq 1 \text{ for all } x.$$

$$R_n(x) \leq \frac{1}{(n+1)!}(0.3)^{n+1} < 0.001$$

By trial and error,  $n = 3$ .

$$50. f(x) = \cos x$$

$$|f^{(n+1)}(x)| \leq 1 \text{ for all } x \text{ and all } n.$$

$$|R_n(x)| = \left| \frac{f^{(n+1)}(z)x^{n+1}}{(n+1)!} \right|$$

$$\leq \frac{(0.1)^{n+1}}{(n+1)!} < 0.001$$

By trial and error,  $n = 2$ .

$$51. f(x) = e^x$$

$$f^{(n+1)}(x) = e^x$$

Max on  $[0, 0.6]$  is  $e^{0.6} \approx 1.8221$ .

$$R_n \leq \frac{1.8221}{(n+1)!}(0.6)^{n+1} < 0.001$$

By trial and error,  $n = 5$ .

52.  $f(x) = e^x$

$$f^{(n+1)}(x) = e^x$$

The maximum value on  $[0, 0.3]$  is  $e^{0.3} \approx 1.3499$ .

$$|R_n| \leq \frac{1.3499}{(n+1)!} (0.3)^{n+1} < 0.001$$

By trial and error,  $n = 3$ .

54.  $f(x) = \cos(\pi x^2)$

$$g(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$f(x) = g(\pi x^2)$$

$$= 1 - \frac{(\pi x^2)^2}{2!} + \frac{(\pi x^2)^4}{4!} - \frac{(\pi x^2)^6}{6!} + \dots$$

$$= 1 - \frac{\pi^2 x^4}{2!} + \frac{\pi^4 x^8}{4!} - \frac{\pi^6 x^{12}}{6!} + \dots$$

$$f(0.6) = 1 - \frac{\pi^2}{2!} (0.6)^4 + \frac{\pi^4}{4!} (0.6)^8 - \frac{\pi^6}{6!} (0.6)^{12} + \dots$$

Since this is an alternating series,

$$R_n \leq a_{n+1} = \frac{\pi^{2n}}{(2n)!} (0.6)^{4n} < 0.0001.$$

By trial and error,  $n = 4$ . Using 4 terms  $f(0.6) \approx 0.4257$ .

56.  $f(x) = e^{-x}$

$$f'(x) = -e^{-x}$$

$$f^{(n+1)}(x) = (-1)^{n+1} e^{-x} \leq 1 \text{ on } [0, 1]$$

$$|R_n| \leq \frac{1}{(n+1)!} 1^{n+1} \leq 0.0001$$

By trial and error,  $n = 7$ .

58.  $f(x) = \sin x \approx x - \frac{x^3}{3!}$

$$|R_3(x)| = \left| \frac{\sin z}{4!} x^4 \right| \leq \frac{|x^4|}{4!} < 0.001$$

$$x^4 < 0.024$$

$$|x| < 0.3936$$

$$-0.3936 < x < 0.3936$$

53.  $f(x) = \ln(x+1)$

$$f^{(n+1)}(x) = \frac{(-1)^n n!}{(x+1)^{n+1}} \Rightarrow \text{Max on } [0, 0.5] \text{ is } n!$$

$$R_n \leq \frac{n!}{(n+1)!} (0.5)^{n+1} = \frac{(0.5)^{n+1}}{n+1} < 0.0001$$

By trial and error,  $n = 9$ . (See Example 9.) Using 9 terms,  $\ln(1.5) \approx 0.4055$ .

55.  $f(x) = e^{-\pi x}$ ,  $f(1.3)$

$$f'(x) = (-\pi)e^{-\pi x}$$

$$f^{(n+1)}(x) = (-\pi)^{n+1} e^{-\pi x} \leq |(-\pi)^{n+1}| \text{ on } [0, 1.3]$$

$$|R_n| \leq \frac{(\pi)^{n+1}}{(n+1)!} (1.3)^{n+1} < 0.0001$$

By trial and error,  $n = 16$ .

57.  $f(x) = e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ ,  $x < 0$

$$R_3(x) = \frac{e^z}{4!} x^4 < 0.001$$

$$e^z x^4 < 0.024$$

$$|xe^{z/4}| < 0.3936$$

$$|x| < \frac{0.3936}{e^{z/4}} < 0.3936, z < 0$$

$$-0.3936 < x < 0$$

59.  $f(x) = \cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ , fifth degree polynomial

$$|f^{(n+1)}(x)| \leq 1 \text{ for all } x \text{ and all } n.$$

$$|R_5(x)| \leq \frac{1}{6!} |x|^6 < 0.001$$

$$|x|^6 < 0.72$$

$$|x| < 0.9467$$

$$-0.9467 < x < 0.9467$$

**Note:** Use a graphing utility to graph  $y = \cos x - (1 - x^2/2 + x^4/24)$  in the viewing window  $[-0.9467, 0.9467] \times [-0.001, 0.001]$  to verify the answer.

$$60. f(x) = e^{-2x} \approx 1 - 2x + 2x^2 - \frac{4}{3}x^3$$

$$f'(x) = -2e^{-2x}, f''(x) = 4e^{-2x}, f'''(x) = -8e^{-2x}, f^{(4)}(x) = 16e^{-2x}$$

$$R_3(x) = \frac{f^{(4)}(z)}{4!}(x-0)^4 = \frac{16e^{-2z}}{24}x^4 = \frac{2}{3}e^{-2z}x^4 < 0.001$$

$$e^{-2z}x^4 < 0.0015$$

$$x < \left[ \frac{0.0015}{e^{-2z}} \right]^{1/4} \approx 0.1970e^{2z} < 0.1970, \text{ for } z < 0.$$

Thus  $0 < x < 0.1970$ .

In fact, by graphing  $f(x) = e^{-2x}$  and  $y = 1 - 2x + 2x^2 - \frac{4}{3}x^3$ , you can verify that  $|f(x) - y| < 0.001$  on  $(-0.19294, 0.20068)$ .

61. The graph of the approximating polynomial  $P$  and the elementary function  $f$  both pass through the point  $(c, f(c))$  and the slopes of  $P$  and  $f$  agree at  $(c, f(c))$ . Depending on the degree of  $P$ , the  $n$ th derivatives of  $P$  and  $f$  agree at  $(c, f(c))$ .

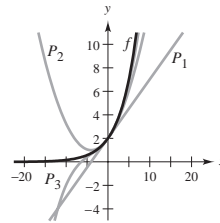
62.  $f(c) = P_2(c)$ ,  $f'(c) = P_2'(c)$ , and  $f''(c) = P_2''(c)$

63. See definition on page 650.

64. See Theorem 9.19, page 654.

65. The accuracy increases as the degree increases (for values within the interval of convergence).

66.



67. (a)  $f(x) = e^x$

$$P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

$$g(x) = xe^x$$

$$Q_5(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5$$

$$Q_5(x) = xP_4(x)$$

(b)  $f(x) = \sin x$

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$g(x) = x \sin x$$

$$Q_6(x) = xP_5(x) = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!}$$

(c)  $g(x) = \frac{\sin x}{x} = \frac{1}{x}P_5(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!}$

68. (a)  $P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$  for  $f(x) = \sin x$

$$P_5'(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

This is the Maclaurin polynomial of degree 4 for  $g(x) = \cos x$ .

(b)  $Q_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$  for  $\cos x$

$$Q_6'(x) = -x + \frac{x^3}{3!} - \frac{x^5}{5!} = -P_5(x)$$

(c)  $R(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$$R'(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

The first four terms are the same!

69. (a)  $Q_2(x) = -1 + \frac{\pi^2(x+2)^2}{32}$       (b)  $R_2(x) = -1 + \frac{\pi^2(x-6)^2}{32}$       (c) No. The polynomial will be linear. Translations are possible at  $x = -2 + 8n$ .

70. Let  $f$  be an odd function and  $P_n$  be the  $n^{\text{th}}$  Maclaurin polynomial for  $f$ . Since  $f$  is odd,  $f'$  is even:

$$f'(-x) = \lim_{h \rightarrow 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \rightarrow 0} \frac{-f(x-h) + f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x+(-h)) - f(x)}{-h} = f'(x).$$

Similarly,  $f''$  is odd,  $f'''$  is even, etc. Therefore,  $f, f'', f^{(4)}$ , etc. are all odd functions, which implies that  $f(0) = f''(0) = \dots = 0$ . Hence, in the formula

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \dots \text{ all the coefficients of the even power of } x \text{ are zero.}$$

71. Let  $f$  be an even function and  $P_n$  be the  $n^{\text{th}}$  Maclaurin polynomial for  $f$ . Since  $f$  is even,  $f'$  is odd,  $f''$  is even,  $f'''$  is odd, etc. All of the odd derivatives of  $f$  are odd and thus, all of the odd powers of  $x$  will have coefficients of zero.  $P_n$  will only have terms with even powers of  $x$ .

72. Let  $P_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots + a_n(x-c)^n$  where  $a_i = \frac{f^{(i)}(c)}{i!}$ .

$$P_n(c) = a_0 = f(c)$$

$$\text{For } 1 \leq k \leq n, \quad P_n^{(k)}(c) = a_n k! = \left( \frac{f^{(k)}(c)}{k!} \right) k! = f^{(k)}(c).$$

73. As you move away from  $x = c$ , the Taylor Polynomial becomes less and less accurate.

## Section 9.8 Power Series

1. Centered at 0                      2. Centered at 0                      3. Centered at 2                      4. Centered at  $\pi$

5.  $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$

$$L = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n x^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{n+1}{n+2} \right| |x| = |x|$$

$$|x| < 1 \Rightarrow R = 1$$

6.  $\sum_{n=0}^{\infty} (2x)^n$

$$L = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x)^{n+1}}{(2x)^n} \right| = 2|x|$$

$$2|x| < 1 \Rightarrow R = \frac{1}{2}$$

7.  $\sum_{n=1}^{\infty} \frac{(2x)^n}{n^2}$

$$L = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(2x)^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{2n^2 x}{(n+1)^2} \right| = 2|x|$$

$$2|x| < 1 \Rightarrow R = \frac{1}{2}$$

8.  $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n}$

$$L = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(-1)^n x^n} \right|$$

$$= \frac{1}{2} |x|$$

$$\frac{1}{2} |x| < 1 \Rightarrow R = 2$$

$$9. \sum_{n=0}^{\infty} \frac{(2x)^{2n}}{(2n)!}$$

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x)^{2n+2}/(2n+2)!}{(2x)^{2n}/(2n)!} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(2x)^2}{(2n+2)(2n+1)} \right| = 0 \end{aligned}$$

Thus, the series converges for all  $x$ .  $R = \infty$ .

$$11. \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

Since the series is geometric, it converges only if  $|x/2| < 1$  or  $-2 < x < 2$ .

$$13. \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{n+1} \cdot \frac{n}{(-1)^n x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{nx}{n+1} \right| = |x| \end{aligned}$$

Interval:  $-1 < x < 1$

When  $x = 1$ , the alternating series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  converges.

When  $x = -1$ , the  $p$ -series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges.

Therefore, the interval of convergence is  $-1 < x \leq 1$ .

$$15. \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right| = 0 \end{aligned}$$

The series converges for all  $x$ . Therefore, the interval of convergence is  $-\infty < x < \infty$ .

$$17. \sum_{n=0}^{\infty} (2n)! \left(\frac{x}{2}\right)^n$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)! x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(2n)! x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)(2n+1)x}{2} \right| = \infty$$

Therefore, the series converges only for  $x = 0$ .

$$10. \sum_{n=0}^{\infty} \frac{(2n)! x^{2n}}{n!}$$

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)! x^{2n+2}/(n+1)!}{(2n)! x^{2n}/n!} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(2n+2)(2n+1)x^2}{(n+1)} \right| = \infty \end{aligned}$$

The series only converges at  $x = 0$ .  $R = 0$ .

$$12. \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n$$

is geometric, so it converges if  $|x/5| < 1 \Rightarrow |x| < 5 \Rightarrow -5 < x < 5$ .

$$14. \sum_{n=0}^{\infty} (-1)^{n+1} (n+1) x^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2} (n+2) x^{n+1}}{(-1)^n (n+1) x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+2)x}{n+1} \right| = |x| \end{aligned}$$

Interval:  $-1 < x < 1$

When  $x = 1$ , the series  $\sum_{n=0}^{\infty} (-1)^{n+1} (n+1)$  diverges.

When  $x = -1$ , the series  $\sum_{n=0}^{\infty} -(n+1)$  diverges.

Therefore, the interval of convergence is  $-1 < x < 1$ .

$$16. \sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(3x)^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{(3x)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{3x}{(2n+2)(2n+1)} \right| = 0 \end{aligned}$$

Therefore, the interval of convergence is  $-\infty < x < \infty$ .

$$18. \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)(n+2)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(n+2)(n+3)} \cdot \frac{(n+1)(n+2)}{(-1)^n x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)x}{n+3} \right| = |x|$$

Interval:  $-1 < x < 1$

When  $x = 1$ , the alternating series  $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n+2)}$  converges.

When  $x = -1$ , the series  $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}$  converges by limit comparison to  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ .

Therefore, the interval of convergence is  $-1 \leq x \leq 1$ .

$$19. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{4^n}$$

Since the series is geometric, it converges only if  $|x/4| < 1$  or  $-4 < x < 4$ .

$$20. \sum_{n=0}^{\infty} \frac{(-1)^n n!(x-4)^n}{3^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (n+1)! (x-4)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(-1)^n n! (x-4)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x-4)}{3} \right| = \infty$$

$R = 0$

Center:  $x = 4$

Therefore, the series converges only for  $x = 4$ .

$$21. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-5)^n}{n5^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2} (x-5)^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n5^n}{(-1)^{n+1} (x-5)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n(x-5)}{5(n+1)} \right| = \frac{1}{5} |x-5|$$

$R = 5$

Center:  $x = 5$

Interval:  $-5 < x - 5 < 5$  or  $0 < x < 10$

When  $x = 0$ , the  $p$ -series  $\sum_{n=1}^{\infty} \frac{-1}{n}$  diverges. When  $x = 10$ , the alternating series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  converges.

Therefore, the interval of convergence is  $0 < x \leq 10$ .

$$22. \sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{(n+1)4^{n+1}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+2}}{(n+2)4^{n+2}} \cdot \frac{(n+1)4^{n+1}}{(x-2)^{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-2)(n+1)}{4(n+2)} \right| = \frac{1}{4} |x-2|$$

$R = 4$

Center:  $x = 2$

Interval:  $-4 < x - 2 < 4$  or  $-2 < x < 6$

When  $x = -2$ , the alternating series  $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)}$  converges.

When  $x = 6$ , the series  $\sum_{n=0}^{\infty} \frac{1}{n+1}$  diverges.

Therefore, the interval of convergence is  $-2 \leq x < 6$ .

$$23. \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(x-1)^{n+2}}{n+2} \cdot \frac{n+1}{(-1)^{n+1}(x-1)^{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x-1)}{n+2} \right| = |x-1|$$

$$R = 1$$

$$\text{Center: } x = 1$$

$$\text{Interval: } -1 < x - 1 < 1 \text{ or } 0 < x < 2$$

When  $x = 0$ , the series  $\sum_{n=0}^{\infty} \frac{1}{n+1}$  diverges by the integral test.

When  $x = 2$ , the alternating series  $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$  converges.

Therefore, the interval of convergence is  $0 < x \leq 2$ .

$$24. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(x-2)^{n+1}}{(n+1)2^{n+1}} \bigg/ \frac{(-1)^{n+1}(x-2)^n}{n2^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x-2}{2} \cdot \frac{n}{n+1} \right| = \left| \frac{x-2}{2} \right| \end{aligned}$$

$$\left| \frac{x-2}{2} \right| < 1 \Rightarrow -2 < x-2 < 2 \Rightarrow 0 < x < 4$$

At  $x = 0$ ,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)(2^n)}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)}{n}, \text{ diverges.}$$

At  $x = 4$ ,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}, \text{ converges.}$$

Interval of convergence:  $0 < x \leq 4$

$$26. \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}x^{2n+3}}{(2n+3)} \cdot \frac{(2n+1)}{(-1)^n x^{2n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+1)}{(2n+3)} x^2 \right| = |x^2|$$

$$R = 1$$

$$\text{Interval: } -1 < x < 1$$

When  $x = 1$ ,  $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$  converges.

When  $x = -1$ ,  $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$  converges.

Therefore, the interval of convergence is  $-1 \leq x \leq 1$ .

$$25. \sum_{n=1}^{\infty} \left( \frac{x-3}{3} \right)^{n-1} \text{ is geometric. It converges if}$$

$$\left| \frac{x-3}{3} \right| < 1 \Rightarrow |x-3| < 3 \Rightarrow 0 < x < 6.$$

Interval convergence:  $0 < x < 6$



$$27. \sum_{n=1}^{\infty} \frac{n}{n+1} (-2x)^{n-1}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)(-2x)^n}{n+2} \cdot \frac{n+1}{n(-2x)^{n-1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(-2x)(n+1)^2}{n(n+2)} \right| = 2|x| \end{aligned}$$

$$R = \frac{1}{2}$$

$$\text{Interval: } -\frac{1}{2} < x < \frac{1}{2}$$

When  $x = -\frac{1}{2}$ , the series  $\sum_{n=1}^{\infty} \frac{n}{n+1}$  diverges by the  $n$ th Term Test.

When  $x = \frac{1}{2}$ , the alternating series  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n+1}$  diverges.

Therefore, the interval of convergence is  $-\frac{1}{2} < x < \frac{1}{2}$ .

$$29. \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x^2}{(2n+2)(2n+3)} \right| = 0 \end{aligned}$$

Therefore, the interval of convergence is  $-\infty < x < \infty$ .

$$31. \sum_{n=1}^{\infty} \frac{2 \cdot 3 \cdot 4 \cdots (n+1)x^n}{n!} = \sum_{n=1}^{\infty} (n+1)x^n$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+2)x^{n+1}}{(n+1)x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n+2}{n+1}x \right| = |x|$$

Converges if  $|x| < 1 \Rightarrow -1 < x < 1$ .

At  $x = \pm 1$ , diverges.

Interval of convergence:  $-1 < x < 1$

$$32. \sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)} (x^{2n+1})$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2 \cdot 4 \cdots (2n)(2n+2)x^{2n+3}}{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)} \cdot \frac{3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdots (2n)x^{2n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)x^2}{(2n+3)} \right| = |x^2|$$

$$R = 1$$

When  $x = \pm 1$ , the series diverges by comparing it to

$$\sum_{n=1}^{\infty} \frac{1}{2n+1}$$

which diverges. Therefore, the interval of convergence is  $-1 < x < 1$ .

$$28. \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(n+1)!} \cdot \frac{n!}{(-1)^n x^{2n}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x^2}{n+1} \right| = 0 \end{aligned}$$

Therefore, the interval of convergence is  $-\infty < x < \infty$ .

$$30. \sum_{n=1}^{\infty} \frac{n!x^n}{(2n)!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!x^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n!x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)x}{(2n+2)(2n+1)} \right| = 0 \end{aligned}$$

Therefore, the interval of convergence is  $-\infty < x < \infty$ .

$$33. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3 \cdot 7 \cdot 11 \cdots (4n-1)(x-3)^n}{4^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2} \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)(4n+3)(x-3)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{(-1)^{n+1} \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)(x-3)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(4n+3)(x-3)}{4} \right| = \infty \end{aligned}$$

$$R = 0$$

$$\text{Center: } x = 3$$

Therefore, the series converges only for  $x = 3$ .

$$34. \sum_{n=1}^{\infty} \frac{n!(x+1)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!(x+1)^{n+1}}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} \bigg/ \frac{(n)!(x+1)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x+1)}{2n+1} \right| = \frac{1}{2}|x+1| \end{aligned}$$

Converges if  $\frac{1}{2}|x+1| < 1 \Rightarrow -2 < x+1 < 2 \Rightarrow -3 < x < 1$ .

$$\text{At } x = 1, a_n = \frac{n!2^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} > 1, \text{ diverges.}$$

$$\text{At } x = -3, a_n = \frac{n!(-2)^n}{1 \cdot 3 \cdots (2n-1)} = (-1)^n \frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n-1)}, \text{ diverges.}$$

Interval of convergence:  $-3 < x < 1$

$$35. \sum_{n=1}^{\infty} \frac{(x-c)^{n-1}}{c^{n-1}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-c)^n}{c^n} \cdot \frac{c^{n-1}}{(x-c)^{n-1}} \right| = \frac{1}{c}|x-c|$$

$$R = c$$

$$\text{Center: } x = c$$

$$\text{Interval: } -c < x - c < c \text{ or } 0 < x < 2c$$

When  $x = 0$ , the series  $\sum_{n=1}^{\infty} (-1)^{n-1}$  diverges.

When  $x = 2c$ , the series  $\sum_{n=1}^{\infty} 1$  diverges.

Therefore, the interval of convergence is  $0 < x < 2c$ .

$$36. \sum_{n=0}^{\infty} \frac{(n!)^k x^n}{(kn)!}, k \text{ is a positive integer.}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{[(n+1)!]^k x^{n+1}}{[k(n+1)]!} \bigg/ \frac{(n!)^k x^n}{(kn)!} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^k x}{(k+nk)(k-1+nk) \cdots (1+nk)} \right| \\ &= \frac{|x|}{k^k} \end{aligned}$$

Converges if  $\frac{|x|}{k^k} < 1 \Rightarrow R = k^k$ .

$$37. \sum_{n=0}^{\infty} \left( \frac{x}{k} \right)^n$$

Since the series is geometric, it converges only if  $|x/k| < 1$  or  $-k < x < k$ .

$$38. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-c)^n}{nc^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(x-c)^{n+1}}{(n+1)c^{n+1}} \cdot \frac{nc^n}{(-1)^{n+1}(x-c)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n(x-c)}{c(n+1)} \right| = \frac{1}{c}|x-c|$$

$$R = c$$

$$\text{Center: } x = c$$

$$\text{Interval: } -c < x - c < c \text{ or } 0 < x < 2c$$

When  $x = 0$ , the  $p$ -series  $\sum_{n=1}^{\infty} \frac{-1}{n}$  diverges.

When  $x = 2c$ , the alternating series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  converges. Therefore, the interval of convergence is  $0 < x \leq 2c$ .

$$39. \sum_{n=1}^{\infty} \frac{k(k+1) \cdots (k+n-1)x^n}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{k(k+1) \cdots (k+n-1)(k+n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k+1) \cdots (k+n-1)x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(k+n)x}{n+1} \right| = |x|$$

$$R = 1$$

When  $x = \pm 1$ , the series diverges and the interval of convergence is  $-1 < x < 1$ .

$$\left[ \frac{k(k+1) \cdots (k+n-1)}{1 \cdot 2 \cdots n} \geq 1 \right]$$

$$40. \sum_{n=1}^{\infty} \frac{n!(x-c)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!(x-c)^{n+1}}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!(x-c)} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x-c)}{2n+1} \right| = \frac{1}{2}|x-c|$$

$$R = 2$$

$$\text{Interval: } -2 < x - c < 2 \text{ or } c - 2 < x < c + 2$$

The series diverges at the endpoints. Therefore, the interval of convergence is  $c - 2 < x < c + 2$ .

$$\left[ \frac{n!(c+2-c)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{n!2^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} > 1 \right]$$

$$41. \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1} + \frac{x^2}{2} + \cdots = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

$$42. \sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n = \sum_{n=1}^{\infty} (-1)^n(n)x^{n-1}$$

$$43. \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$$

Replace  $n$  with  $n - 1$ .

$$44. \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{2n-1}$$

$$45. \text{(a) } f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n, -2 < x < 2 \quad (\text{Geometric})$$

$$\text{(b) } f'(x) = \sum_{n=1}^{\infty} \left(\frac{n}{2}\right) \left(\frac{x}{2}\right)^{n-1}, -2 < x < 2$$

$$\text{(c) } f''(x) = \sum_{n=2}^{\infty} \left(\frac{n}{2}\right) \left(\frac{n-1}{2}\right) \left(\frac{x}{2}\right)^{n-2}, -2 < x < 2$$

$$\text{(d) } \int f(x) dx = \sum_{n=0}^{\infty} \frac{2}{n+1} \left(\frac{x}{2}\right)^{n+1}, -2 \leq x < 2$$

$$46. \text{(a) } f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}, 0 < x \leq 10$$

$$\text{(b) } f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^{n-1}}{5^n}, 0 < x < 10$$

$$\text{(c) } f''(x) = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}(n-1)(x-5)^{n-2}}{5^n}, 0 < x < 10$$

$$\text{(d) } \int f(x) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^{n+1}}{n(n+1)5^n}, 0 \leq x \leq 10$$

47. (a)  $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}, 0 < x \leq 2$   
 (b)  $f'(x) = \sum_{n=0}^{\infty} (-1)^{n+1}(x-1)^n, 0 < x < 2$   
 (c)  $f''(x) = \sum_{n=1}^{\infty} (-1)^{n+1}n(x-1)^{n-1}, 0 < x < 2$   
 (d)  $\int f(x) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+2}}{(n+1)(n+2)}, 0 \leq x \leq 2$

48. (a)  $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n}, 1 < x \leq 3$   
 (b)  $f'(x) = \sum_{n=1}^{\infty} (-1)^{n+1}(x-2)^{n-1}, 1 < x < 3$   
 (c)  $f''(x) = \sum_{n=2}^{\infty} (-1)^{n+1}(n-1)(x-2)^{n-2}, 1 < x < 3$   
 (d)  $\int f(x) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^{n+1}}{n(n+1)}, 1 \leq x \leq 3$

49.  $g(1) = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = 1 + \frac{1}{3} + \frac{1}{9} + \dots$   
 $S_1 = 1, S_2 = 1.33$ . Matches (c).

50.  $g(2) = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \frac{4}{9} + \dots$   
 $S_1 = 1, S_2 = 1.67$ . Matches (a).

51.  $g(3.1) = \sum_{n=0}^{\infty} \left(\frac{3.1}{3}\right)^n$  diverges. Matches (b).

52.  $g(-2) = \sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n$  alternating. Matches (d).

53.  $g\left(\frac{1}{8}\right) = \sum_{n=0}^{\infty} \left[2\left(\frac{1}{8}\right)\right]^n = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = 1 + \frac{1}{4} + \frac{1}{16} + \dots$ , converges  
 $S_1 = 1, S_2 = 1.25, S_3 = 1.3125$  Matches (b).

54.  $g\left(-\frac{1}{8}\right) = \sum_{n=0}^{\infty} \left[2\left(-\frac{1}{8}\right)\right]^n = \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n = 1 - \frac{1}{4} + \frac{1}{16} - \dots$ , converges  
 $S_1 = 1, S_2 = 0.75, S_3 = 0.8125$  Matches (c).

55.  $g\left(\frac{9}{16}\right) = \sum_{n=0}^{\infty} \left[2\left(\frac{9}{16}\right)\right]^n = \sum_{n=0}^{\infty} \left(\frac{9}{8}\right)^n$ , diverges  
 $S_1 = 1, S_2 = \frac{17}{8}$  Matches (d).

56.  $g\left(\frac{3}{8}\right) = \sum_{n=0}^{\infty} \left[2\left(\frac{3}{8}\right)\right]^n = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$ , converges  
 $S_1 = 1, S_2 = 1.75$  Matches (a).

57. A series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

is called a power series centered at  $c$ .

58. The set of all values of  $x$  for which the power series converges is the interval of convergence. If the power series converges for all  $x$ , then the radius of convergence is  $R = \infty$ . If the power series converges at only  $c$ , then  $R = 0$ . Otherwise, according to Theorem 8.20, there exists a real number  $R > 0$  (radius of convergence) such that the series converges absolutely for  $|x - c| < R$  and diverges for  $|x - c| > R$ .

59. A single point, an interval, or the entire real line.

60. You differentiate and integrate the power series term by term. The radius of convergence remains the same. However, the interval of convergence might change.

61. Answers will vary.

$\sum_{n=1}^{\infty} \frac{x^n}{n}$  converges for  $-1 \leq x < 1$ . At  $x = -1$ , the convergence is conditional because  $\sum \frac{1}{n}$  diverges.

$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$  converges for  $-1 \leq x \leq 1$ . At  $x = \pm 1$ , the convergence is absolute.

62. Many answers possible.

(a)  $\sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n$  Geometric:  $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$

(b)  $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$  converges for  $-1 < x \leq 1$

(c)  $\sum_{n=1}^{\infty} (2x+1)^n$  Geometric:  $|2x+1| < 1 \Rightarrow -1 < x < 0$

(d)  $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n4^n}$  converges for  $-2 \leq x < 6$

63. (a)  $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ ,  $-\infty < x < \infty$

(See Exercise 29.)

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad -\infty < x < \infty$$

(b)  $f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = g(x)$

(c)  $g'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}$   
 $= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = -f(x)$

(d)  $f(x) = \sin x$  and  $g(x) = \cos x$

64. (a)  $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ ,  $-\infty < x < \infty$  (See Exercise 11)

(b)  $f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$

(c)  $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$f(0) = 1$

(d)  $f(x) = e^x$

65.  $y = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}$   
 $y' = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$   
 $y'' = \sum_{n=1}^{\infty} \frac{(-1)^n (2n)x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)x^{2n-1}}{(2n-1)!}$   
 $y'' + y = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} = 0$

66.  $y = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-2}}{(2n-2)!}$   
 $y' = \sum_{n=1}^{\infty} \frac{(-1)^n (2n)x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}$   
 $y'' = \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)x^{2n-2}}{(2n-1)!} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-2}}{(2n-2)!}$   
 $y'' + y = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-2}}{(2n-2)!} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-2}}{(2n-2)!} = 0$

67.  $y = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$

$y' = \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

$y'' = \sum_{n=1}^{\infty} \frac{(2n)x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!} = y$

$y'' - y = 0$

68.  $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{x^{2n-2}}{(2n-2)!}$

$y' = \sum_{n=1}^{\infty} \frac{(2n-2)x^{2n-1}}{(2n-2)!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$

$y'' = \sum_{n=1}^{\infty} \frac{(2n-1)x^{2n-2}}{(2n-1)!} = \sum_{n=1}^{\infty} \frac{x^{2n-2}}{(2n-2)!} = y$

$y'' - y = 0$

69.  $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$   $y' = \sum_{n=1}^{\infty} \frac{2nx^{2n-1}}{2^n n!}$   $y'' = \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!}$

$$y'' - xy' - y = \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!} - \sum_{n=1}^{\infty} \frac{2nx^{2n}}{2^n n!} - \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$$
$$= \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!} - \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{2^n n!}$$
$$= \sum_{n=0}^{\infty} \left[ \frac{(2n+2)(2n+1)x^{2n}}{2^{n+1}(n+1)!} - \frac{(2n+1)x^{2n}}{2^n n!} \cdot \frac{2(n+1)}{2(n+1)} \right]$$
$$= \sum_{n=0}^{\infty} \frac{2(n+1)x^{2n} [(2n+1) - (2n+1)]}{2^{n+1}(n+1)!} = 0$$

$$\begin{aligned}
 70. \quad y &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdot \dots \cdot (4n-1)} \\
 y' &= \sum_{n=1}^{\infty} \frac{(-1)^n 4nx^{4n-1}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdot \dots \cdot (4n-1)} \\
 y'' &= \sum_{n=1}^{\infty} \frac{(-1)^n 4n(4n-1)x^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdot \dots \cdot (4n-1)} = -x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n 4nx^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdot \dots \cdot (4n-5)} \\
 y'' + x^2 y &= -x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n 4nx^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdot \dots \cdot (4n-5)} + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n+2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdot \dots \cdot (4n-1)} + x^2 \\
 &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4(n+1)x^{4n+2}}{2^{2n+2}(n+1)! \cdot 3 \cdot 7 \cdot 11 \cdot \dots \cdot (4n-1)} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n+2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdot \dots \cdot (4n-1)} \frac{2^2(n+1)}{2^2(n+1)} = 0
 \end{aligned}$$

$$71. J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} (k!)^2}$$

$$(a) \lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+1} x^{2k+2}}{2^{2k+2} [(k+1)!]^2} \cdot \frac{2^{2k} (k!)^2}{(-1)^k x^{2k}} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)x^2}{2^2(k+1)^2} \right| = 0$$

Therefore, the interval of convergence is  $-\infty < x < \infty$ .

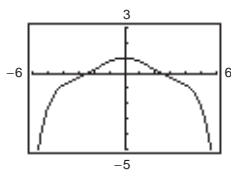
$$(b) \quad J_0 = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{4^k (k!)^2}$$

$$J_0' = \sum_{k=1}^{\infty} (-1)^k \frac{2kx^{2k-1}}{4^k (k!)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(2k+2)x^{2k+1}}{4^{k+1} [(k+1)!]^2}$$

$$J_0'' = \sum_{k=1}^{\infty} (-1)^k \frac{2k(2k-1)x^{2k-2}}{4^k (k!)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(2k+2)(2k+1)x^{2k}}{4^{k+1} [(k+1)!]^2}$$

$$\begin{aligned}
 x^2 J_0'' + x J_0' + x^2 J_0 &= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2(2k+1)x^{2k+2}}{4^{k+1}(k+1)!k!} + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2x^{2k+2}}{4^{k+1}(k+1)!k!} + \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+2}}{4^k (k!)^2} \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{4^k (k!)^2} \left[ (-1) \frac{2(2k+1)}{4(k+1)} + (-1) \frac{2}{4(k+1)} + 1 \right] \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{4^k (k!)^2} \left[ \frac{-4k-2}{4k+4} - \frac{2}{4k+4} + \frac{4k+4}{4k+4} \right] = 0
 \end{aligned}$$

$$(c) P_6(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304}$$



$$\begin{aligned}
 (d) \int_0^1 J_0 dx &= \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{4^k (k!)^2} dx \\
 &= \left[ \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{4^k (k!)^2 (2k+1)} \right]_0^1 \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k (k!)^2 (2k+1)} \\
 &= 1 - \frac{1}{12} + \frac{1}{320} \approx 0.92
 \end{aligned}$$

(integral is approximately 0.9197304101)

$$72. J_1(x) = x \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+1} k! (k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k! (k+1)!}$$

$$(a) \lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+1} x^{2k+3}}{2^{2k+3} (k+1)! (k+2)!} \cdot \frac{2^{2k+1} k! (k+1)!}{(-1)^k x^{2k+1}} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)x^2}{2^2 (k+2)(k+1)} \right| = 0$$

Therefore, the interval of convergence is  $-\infty < x < \infty$ .

—CONTINUED—

72. —CONTINUED

$$(b) J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k!(k+1)!}$$

$$J_1'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)x^{2k}}{2^{2k+1} k!(k+1)!}$$

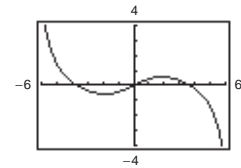
$$J_1''(x) = \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k)x^{2k-1}}{2^{2k+1} k!(k+1)!}$$

$$\begin{aligned} x^2 J_1'' + x J_1' + (x^2 - 1) J_1 &= \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k)x^{2k+1}}{2^{2k+1} k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)x^{2k+1}}{2^{2k+1} k!(k+1)!} \\ &\quad + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} - \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k!(k+1)!} \\ &= \left[ \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k)x^{2k+1}}{2^{2k+1} k!(k+1)!} + \frac{x}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)x^{2k+1}}{2^{2k+1} k!(k+1)!} \right. \\ &\quad \left. - \frac{x}{2} - \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k!(k+1)!} \right] + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1} [(2k+1)(2k) + (2k+1) - 1]}{2^{2k+1} k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1} 4k(k+1)}{2^{2k+1} k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k-1} (k-1)! k!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+3}}{2^{2k+1} k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3} [(-1) + 1]}{2^{2k+1} k!(k+1)!} = 0 \end{aligned}$$

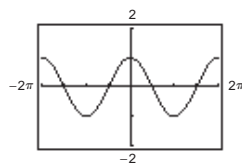
$$(c) P_7(x) = \frac{x}{2} - \frac{1}{16} x^3 + \frac{1}{384} x^5 - \frac{1}{18,432} x^7$$

$$(d) J_0'(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} 2(k+1)x^{2k+1}}{2^{2k+2} (k+1)!(k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2^{2k+1} k!(k+1)!}$$

$$-J_1(x) = -\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k!(k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2^{2k+1} k!(k+1)!} \quad \text{Note: } J_0'(x) = -J_1(x)$$

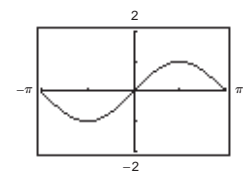


$$73. f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$$



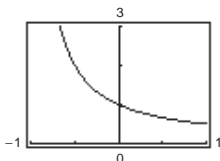
$$74. f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= \sin x$$

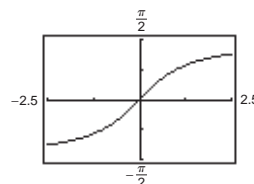


$$75. f(x) = \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n \quad \text{Geometric}$$

$$= \frac{1}{1 - (-x)} = \frac{1}{1+x} \quad \text{for } -1 < x < 1$$

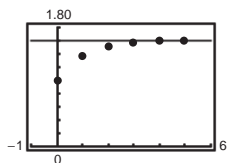


$$76. f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x, \quad -1 \leq x \leq 1$$



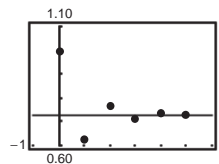
77.  $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$

(a)  $\sum_{n=0}^{\infty} \left(\frac{3/4}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{3}{8}\right)^n = \frac{1}{1 - (3/8)} = \frac{8}{5} = 1.6$



- (c) The alternating series converges more rapidly. The partial sums of the series of positive terms approach the sum from below. The partial sums of the alternating series alternate sides of the horizontal line representing the sum.

(b)  $\sum_{n=0}^{\infty} \left(\frac{-3/4}{2}\right)^n = \sum_{n=0}^{\infty} \left(-\frac{3}{8}\right)^n = \frac{1}{1 - (-3/8)} = \frac{8}{11} \approx 0.7272$

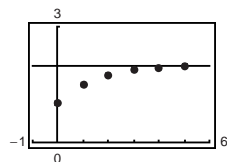


(d)  $\sum_{n=0}^N \left(\frac{3}{2}\right)^n > M$

|     |    |     |      |        |
|-----|----|-----|------|--------|
| $M$ | 10 | 100 | 1000 | 10,000 |
| $N$ | 4  | 9   | 15   | 21     |

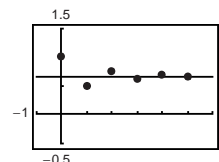
78.  $\sum_{n=0}^{\infty} (3x)^n$  converges on  $\left(-\frac{1}{3}, \frac{1}{3}\right)$ .

(a)  $x = \frac{1}{6}: \sum_{n=0}^{\infty} \left(3\left(\frac{1}{6}\right)\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - (1/2)} = 2$



- (c) The alternating series converges more rapidly. The partial sums in (a) approach the sum 2 from below. The partial sums in (b) alternate sides of the horizontal line  $y = \frac{2}{3}$ .

(b)  $x = -\frac{1}{6}: \sum_{n=0}^{\infty} \left(3\left(-\frac{1}{6}\right)\right)^n = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{1}{1 + (1/2)} = \frac{2}{3}$



(d)  $\sum_{n=0}^N \left(3 \cdot \frac{2}{3}\right)^n = \sum_{n=0}^N 2^n > M$

|     |    |     |      |        |
|-----|----|-----|------|--------|
| $M$ | 10 | 100 | 1000 | 10,000 |
| $N$ | 3  | 6   | 9    | 13     |

79. False;

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n 2^n}$$

 converges for  $x = 2$  but diverges for  $x = -2$ .

80. True; if

$$\sum_{n=0}^{\infty} a_n x^n$$

 converges for  $x = 2$ , then we know that it must converge on  $(-2, 2]$ .

 81. True; the radius of convergence is  $R = 1$  for both series.

82. True

$$\int_0^1 f(x) dx = \int_0^1 \left( \sum_{n=0}^{\infty} a_n x^n \right) dx = \left[ \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} \right]_0^1 = \sum_{n=0}^{\infty} \frac{a_n}{n+1}$$



$$83. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1+p)!}{(n+1)!(n+1+q)!} x^{n+1} \Big/ \frac{(n+p)!}{n!(n+q)!} x^n \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1+p)x}{(n+1)(n+1+q)} \right| = 0$$

Thus, the series converges for all  $x$ :  $R = \infty$ .

$$84. (a) g(x) = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots$$

$$S_{2n} = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots + x^{2n} + 2x^{2n+1} \\ = (1 + x^2 + x^4 + \cdots + x^{2n}) + 2x(1 + x^2 + x^4 + \cdots + x^{2n})$$

$$\lim_{n \rightarrow \infty} S_{2n} = \sum_{n=0}^{\infty} x^{2n} + 2x \sum_{n=0}^{\infty} x^{2n}$$

Since each series is geometric,  $R = 1$ .

$$(b) \text{ For } |x| < 1, g(x) = \frac{1}{1-x^2} + 2x \frac{1}{1-x^2} = \frac{1+2x}{1-x^2}.$$

$$85. (a) f(x) = \sum_{n=0}^{\infty} c_n x^n, c_{n+3} = c_n$$

$$= c_0 + c_1 x + c_2 x^2 + c_0 x^3 + c_1 x^4 + c_2 x^5 + c_0 x^6 + \cdots$$

$$S_{3n} = c_0(1 + x^3 + \cdots + x^{3n}) + c_1 x(1 + x^3 + \cdots + x^{3n}) + c_2 x^2(1 + x^3 + \cdots + x^{3n})$$

$$\lim_{n \rightarrow \infty} S_{3n} = c_0 \sum_{n=0}^{\infty} x^{3n} + c_1 x \sum_{n=0}^{\infty} x^{3n} + c_2 x^2 \sum_{n=0}^{\infty} x^{3n}$$

Each series is geometric,  $R = 1$ , and the interval of convergence is  $(-1, 1)$ .

$$(b) \text{ For } |x| < 1, f(x) = c_0 \frac{1}{1-x^3} + c_1 x \frac{1}{1-x^3} + c_2 x^2 \frac{1}{1-x^3} = \frac{c_0 + c_1 x + c_2 x^2}{1-x^3}.$$

86. For the series  $\sum c_n x^n$ ,

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1} x^{n+1}}{c_n x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} x \right| < 1 \Rightarrow |x| < \left| \frac{c_n}{c_{n+1}} \right| = R$$

For the series  $\sum c_n x^{2n}$ ,

$$\lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1} x^{2n+2}}{c_n x^{2n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} x^2 \right| < 1 \Rightarrow |x^2| < \left| \frac{c_n}{c_{n+1}} \right| = R \Rightarrow |x| < \sqrt{R}.$$

87. At  $x_0 + R$ , the series is  $\sum_{n=0}^{\infty} c_n (x_0 - x_0 - R)^n = \sum_{n=0}^{\infty} c_n (-R)^n$  which converges.

At  $x_0 - R$ , the series is  $\sum_{n=0}^{\infty} c_n R^n$ , which diverges.

Hence, at  $x_0 + R$ , we have  $\sum_{n=0}^{\infty} c_n (-R)^n$  converges,  $\sum_{n=0}^{\infty} |c_n (-R)^n| = \sum_{n=0}^{\infty} c_n R^n$  diverges.

$\Rightarrow$  The series converges conditionally.

## Section 9.9 Representation of Functions by Power Series

$$\begin{aligned} 1. \text{ (a) } \frac{1}{2-x} &= \frac{1/2}{1-(x/2)} = \frac{a}{1-r} \\ &= \sum_{n=0}^{\infty} \frac{1}{2} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}} \end{aligned}$$

This series converges on  $(-2, 2)$ .

$$\begin{array}{r} \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \cdots \\ \hline \text{(b) } 2-x \quad 1 \\ 1 - \frac{x}{2} \\ \hline \frac{x}{2} \\ \frac{x}{2} - \frac{x^2}{4} \\ \hline \frac{x^2}{4} \\ \frac{x^2}{4} - \frac{x^3}{8} \\ \hline \frac{x^3}{8} \\ \frac{x^3}{8} - \frac{x^4}{16} \\ \hline \vdots \end{array}$$

$$\begin{aligned} 2. \text{ (a) } f(x) &= \frac{4}{5-x} = \frac{4/5}{1-x/5} = \frac{a}{1-r} \\ &= \sum_{n=0}^{\infty} \frac{4}{5} \left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty} \frac{4x^n}{5^{n+1}} \end{aligned}$$

This series converges on  $(-5, 5)$ .

$$\begin{array}{r} \frac{4}{5} + \frac{4}{25}x + \frac{4}{125}x^2 + \frac{4x^3}{625} + \cdots \\ \hline \text{(b) } 5-x \quad 4 \\ 4 - \frac{4}{5}x \\ \hline \frac{4}{5}x \\ \frac{4}{5}x - \frac{4}{25}x^2 \\ \hline \frac{4}{25}x^2 \\ \frac{4}{25}x^2 - \frac{4x^3}{125} \\ \hline \frac{4x^3}{125} \\ \frac{4x^3}{125} - \frac{4x^4}{625} \\ \hline \vdots \end{array}$$

$$\begin{aligned} 3. \text{ (a) } \frac{1}{2+x} &= \frac{1/2}{1-(-x/2)} = \frac{a}{1-r} \\ &= \sum_{n=0}^{\infty} \frac{1}{2} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}} \end{aligned}$$

This series converges on  $(-2, 2)$ .

$$\begin{array}{r} \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \cdots \\ \hline \text{(b) } 2+x \quad 1 \\ 1 + \frac{x}{2} \\ \hline -\frac{x}{2} \\ -\frac{x}{2} - \frac{x^2}{4} \\ \hline \frac{x^2}{4} \\ \frac{x^2}{4} + \frac{x^3}{8} \\ \hline -\frac{x^3}{8} \\ -\frac{x^3}{8} - \frac{x^4}{16} \\ \hline \vdots \end{array}$$

$$\begin{aligned} 4. \text{ (a) } \frac{1}{1+x} &= \frac{1}{1-(-x)} = \frac{a}{1-r} \\ &= \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n \end{aligned}$$

This series converges on  $(-1, 1)$ .

$$\begin{array}{r} 1 - x + x^2 - x^3 + \cdots \\ \hline \text{(b) } 1+x \quad 1 \\ 1+x \\ \hline -x \\ -x - x^2 \\ \hline x^2 \\ x^2 + x^3 \\ \hline -x^3 \\ -x^3 - x^4 \\ \hline \vdots \end{array}$$

5. Writing  $f(x)$  in the form  $a/(1-r)$ , we have

$$\frac{1}{2-x} = \frac{1}{-3-(x-5)} = \frac{-1/3}{1+(1/3)(x-5)}$$

which implies that  $a = -1/3$  and  $r = (-1/3)(x-5)$ . Therefore, the power series for  $f(x)$  is given by

$$\begin{aligned} \frac{1}{2-x} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} -\frac{1}{3} \left[ -\frac{1}{3}(x-5) \right]^n \\ &= \sum_{n=0}^{\infty} \frac{(x-5)^n}{(-3)^{n+1}}, \quad |x-5| < 3 \text{ or } 2 < x < 8. \end{aligned}$$

7. Writing  $f(x)$  in the form  $a/(1-r)$ , we have

$$\frac{3}{2x-1} = \frac{-3}{1-2x} = \frac{a}{1-r}$$

which implies that  $a = -3$  and  $r = 2x$ . Therefore, the power series for  $f(x)$  is given by

$$\begin{aligned} \frac{3}{2x-1} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} (-3)(2x)^n \\ &= -3 \sum_{n=0}^{\infty} (2x)^n, \quad |2x| < 1 \text{ or } -\frac{1}{2} < x < \frac{1}{2}. \end{aligned}$$

9. Writing  $f(x)$  in the form  $a/(1-r)$ , we have

$$\begin{aligned} \frac{1}{2x-5} &= \frac{-1}{11-2(x+3)} \\ &= \frac{-1/11}{1-(2/11)(x+3)} = \frac{a}{1-r} \end{aligned}$$

which implies that  $a = -1/11$  and  $r = (2/11)(x+3)$ . Therefore, the power series for  $f(x)$  is given by

$$\begin{aligned} \frac{1}{2x-5} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \left( -\frac{1}{11} \right) \left[ \frac{2}{11}(x+3) \right]^n \\ &= -\sum_{n=0}^{\infty} \frac{2^n(x+3)^n}{11^{n+1}}, \end{aligned}$$

$$|x+3| < \frac{11}{2} \text{ or } -\frac{17}{2} < x < \frac{5}{2}.$$

11. Writing  $f(x)$  in the form  $a/(1-r)$ , we have

$$\frac{3}{x+2} = \frac{3}{2+x} = \frac{3/2}{1+(1/2)x} = \frac{a}{1-r}$$

which implies that  $a = 3/2$  and  $r = (-1/2)x$ . Therefore, the power series for  $f(x)$  is given by

$$\begin{aligned} \frac{3}{x+2} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{3}{2} \left( -\frac{1}{2}x \right)^n \\ &= 3 \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}} = \frac{3}{2} \sum_{n=0}^{\infty} \left( -\frac{x}{2} \right)^n, \end{aligned}$$

$$|x| < 2 \text{ or } -2 < x < 2.$$

6. Writing  $f(x)$  in the form  $a/(1-r)$ , we have

$$\frac{4}{5-x} = \frac{4}{7-(x+2)} = \frac{4/7}{1-1/7(x+2)} = \frac{a}{1-r}.$$

Therefore, the power series for  $f(x)$  is given by

$$\begin{aligned} \frac{4}{5-x} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{4}{7} \left( \frac{1}{7}(x+2) \right)^n \\ &= \sum_{n=0}^{\infty} \frac{4(x+2)^n}{7^{n+1}}. \end{aligned}$$

$$|x+2| < 7 \text{ or } -9 < x < 5$$

8. Writing  $f(x)$  in the form  $a/(1-r)$ , we have

$$\frac{3}{2x-1} = \frac{3}{3+2(x-2)} = \frac{1}{1+(2/3)(x-2)} = \frac{a}{1-r}$$

which implies that  $a = 1$  and  $r = (-2/3)(x-2)$ . Therefore, the power series for  $f(x)$  is given by

$$\begin{aligned} \frac{3}{2x-1} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \left[ -\frac{2}{3}(x-2) \right]^n \\ &= \sum_{n=0}^{\infty} \frac{(-2)^n(x-2)^n}{3^n}, \end{aligned}$$

$$|x-2| < \frac{3}{2} \text{ or } \frac{1}{2} < x < \frac{7}{2}.$$

10. Writing  $f(x)$  in the form  $a/(1-r)$ , we have

$$\frac{1}{2x-5} = \frac{1}{-5+2x} = \frac{-1/5}{1-(2/5)x} = \frac{a}{1-r}$$

which implies that  $a = -1/5$  and  $r = (2/5)x$ . Therefore, the power series for  $f(x)$  is given by

$$\frac{1}{2x-5} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \left( -\frac{1}{5} \right) \left( \frac{2}{5}x \right)^n = -\sum_{n=0}^{\infty} \frac{2^n x^n}{5^{n+1}},$$

$$|x| < \frac{5}{2} \text{ or } -\frac{5}{2} < x < \frac{5}{2}.$$

12. Writing  $f(x)$  in the form  $a/(1-r)$ , we have

$$\frac{4}{3x+2} = \frac{4}{8+3(x-2)} = \frac{1/2}{1+(3/8)(x-2)} = \frac{a}{1-r}$$

which implies that  $a = 1/2$  and  $r = (-3/8)(x-2)$ . Therefore, the power series for  $f(x)$  is given by

$$\begin{aligned} \frac{4}{3x+2} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{1}{2} \left[ -\frac{3}{8}(x-2) \right]^n \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-3)^n (x-2)^n}{8^n}, \end{aligned}$$

$$|x-2| < \frac{8}{3} \text{ or } -\frac{2}{3} < x < \frac{14}{3}.$$

$$13. \frac{3x}{x^2 + x - 2} = \frac{2}{x + 2} + \frac{1}{x - 1} = \frac{2}{2 + x} + \frac{1}{-1 + x} = \frac{1}{1 + (1/2)x} + \frac{-1}{1 - x}$$

Writing  $f(x)$  as a sum of two geometric series, we have

$$\frac{3x}{x^2 + x - 2} = \sum_{n=0}^{\infty} \left(-\frac{1}{2}x\right)^n + \sum_{n=0}^{\infty} (-1)(x)^n = \sum_{n=0}^{\infty} \left[\frac{1}{(-2)^n} - 1\right]x^n.$$

The interval of convergence is  $-1 < x < 1$  since

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(1 - (-2)^{n+1})x^{n+1}}{(-2)^{n+1}} \cdot \frac{(-2)^n}{(1 - (-2)^n)x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(1 - (-2)^{n+1})x}{-2 - (-2)^{n+1}} \right| = |x|.$$

$$14. \frac{4x - 7}{2x^2 + 3x - 2} = \frac{3}{x + 2} - \frac{2}{2x - 1} = \frac{3}{2 + x} - \frac{2}{-1 + 2x} = \frac{3/2}{1 + (1/2)x} + \frac{2}{1 - 2x}$$

Writing  $f(x)$  as a sum of two geometric series, we have

$$\frac{4x - 7}{2x^2 + 3x - 2} = \sum_{n=0}^{\infty} \left(\frac{3}{2}\right) \left(-\frac{1}{2}x\right)^n + \sum_{n=0}^{\infty} 2(2x)^n = \sum_{n=0}^{\infty} \left[\frac{3(-1)^n}{2^{n+1}} + 2^{n+1}\right]x^n, \quad |x| < \frac{1}{2} \text{ or } -\frac{1}{2} < x < \frac{1}{2}.$$

$$15. \frac{2}{1 - x^2} = \frac{1}{1 - x} + \frac{1}{1 + x}$$

Writing  $f(x)$  as a sum of two geometric series, we have

$$\frac{2}{1 - x^2} = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (1 + (-1)^n)x^n = \sum_{n=0}^{\infty} 2x^{2n}.$$

The interval of convergence is  $|x^2| < 1$  or  $-1 < x < 1$  since  $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2x^{2n+2}}{2x^{2n}} \right| = |x^2|$ .

16. First finding the power series for  $4/(4 + x)$ , we have

$$\frac{1}{1 + (1/4)x} = \sum_{n=0}^{\infty} \left(-\frac{1}{4}x\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{4^n}$$

Now replace  $x$  with  $x^2$ .

$$\frac{4}{4 + x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n}.$$

The interval of convergence is  $|x^2| < 4$  or  $-2 < x < 2$  since

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{4^{n+1}} \cdot \frac{4^n}{(-1)^n x^{2n}} \right| = \left| -\frac{x^2}{4} \right| = \frac{|x^2|}{4}.$$

$$17. \frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} (-1)^n (-x)^n = \sum_{n=0}^{\infty} (-1)^{2n} x^n = \sum_{n=0}^{\infty} x^n$$

$$h(x) = \frac{-2}{x^2 - 1} = \frac{1}{1 + x} + \frac{1}{1 - x} = \sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} [(-1)^n + 1]x^n$$

$$= 2 + 0x + 2x^2 + 0x^3 + 2x^4 + 0x^5 + 2x^6 + \cdots = \sum_{n=0}^{\infty} 2x^{2n}, \quad -1 < x < 1 \text{ (See Exercise 15.)}$$

$$\begin{aligned}
 18. \quad h(x) &= \frac{x}{x^2 - 1} = \frac{1}{2(1+x)} - \frac{1}{2(1-x)} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n x^n - \frac{1}{2} \sum_{n=0}^{\infty} x^n \\
 &= \frac{1}{2} \sum_{n=0}^{\infty} [(-1)^n - 1] x^n = \frac{1}{2} [0 - 2x + 0x^2 - 2x^3 + 0x^4 - 2x^5 + \cdots] \\
 &= \frac{1}{2} \sum_{n=0}^{\infty} (-2)x^{2n+1} = - \sum_{n=0}^{\infty} x^{2n+1}, \quad -1 < x < 1
 \end{aligned}$$

19. By taking the first derivative, we have  $\frac{d}{dx} \left[ \frac{1}{x+1} \right] = \frac{-1}{(x+1)^2}$ . Therefore,

$$\begin{aligned}
 \frac{-1}{(x+1)^2} &= \frac{d}{dx} \left[ \sum_{n=0}^{\infty} (-1)^n x^n \right] = \sum_{n=1}^{\infty} (-1)^n n x^{n-1} \\
 &= \sum_{n=0}^{\infty} (-1)^{n+1} (n+1) x^n, \quad -1 < x < 1.
 \end{aligned}$$

20. By taking the second derivative, we have  $\frac{d^2}{dx^2} \left[ \frac{1}{x+1} \right] = \frac{2}{(x+1)^3}$ . Therefore,

$$\begin{aligned}
 \frac{2}{(x+1)^3} &= \frac{d^2}{dx^2} \left[ \sum_{n=0}^{\infty} (-1)^n x^n \right] \\
 &= \frac{d}{dx} \left[ \sum_{n=1}^{\infty} (-1)^n n x^{n-1} \right] = \sum_{n=2}^{\infty} (-1)^n n(n-1) x^{n-2} = \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^n, \quad -1 < x < 1.
 \end{aligned}$$

21. By integrating, we have  $\int \frac{1}{x+1} dx = \ln(x+1)$ . Therefore,

$$\ln(x+1) = \int \left[ \sum_{n=0}^{\infty} (-1)^n x^n \right] dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}, \quad -1 < x \leq 1.$$

To solve for  $C$ , let  $x = 0$  and conclude that  $C = 0$ . Therefore,

$$\ln(x+1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}, \quad -1 < x \leq 1.$$

22. By integrating, we have

$$\int \frac{1}{1+x} dx = \ln(1+x) + C_1 \quad \text{and} \quad \int \frac{1}{1-x} dx = -\ln(1-x) + C_2.$$

$f(x) = \ln(1-x^2) = \ln(1+x) - [-\ln(1-x)]$ . Therefore,

$$\begin{aligned}
 \ln(1-x^2) &= \int \frac{1}{1+x} dx - \int \frac{1}{1-x} dx \\
 &= \int \left[ \sum_{n=0}^{\infty} (-1)^n x^n \right] dx - \int \left[ \sum_{n=0}^{\infty} x^n \right] dx = \left[ C_1 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \right] - \left[ C_2 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \right] \\
 &= C + \sum_{n=0}^{\infty} \frac{[(-1)^n - 1] x^{n+1}}{n+1} = C + \sum_{n=0}^{\infty} \frac{-2x^{2n+2}}{2n+2} = C + \sum_{n=0}^{\infty} \frac{(-1)x^{2n+2}}{n+1}
 \end{aligned}$$

To solve for  $C$ , let  $x = 0$  and conclude that  $C = 0$ . Therefore,

$$\ln(1-x^2) = - \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n+1}, \quad -1 < x < 1.$$

$$23. \quad \frac{1}{x^2+1} = \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad -1 < x < 1$$

$$24. \frac{2x}{x^2 + 1} = 2x \sum_{n=0}^{\infty} (-1)^n x^{2n} \text{ (See Exercise 23.)}$$

$$= \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}$$

Since  $\frac{d}{dx}(\ln(x^2 + 1)) = \frac{2x}{x^2 + 1}$ , we have

$$\ln(x^2 + 1) = \int \left[ \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1} \right] dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}, \quad -1 \leq x \leq 1.$$

To solve for  $C$ , let  $x = 0$  and conclude that  $C = 0$ . Therefore,

$$\ln(x^2 + 1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}, \quad -1 \leq x \leq 1.$$

$$25. \text{ Since, } \frac{1}{x+1} = \sum_{n=0}^{\infty} (-1)^n x^n, \text{ we have } \frac{1}{4x^2+1} = \sum_{n=0}^{\infty} (-1)^n (4x^2)^n = \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n} = \sum_{n=0}^{\infty} (-1)^n (2x)^{2n}, \quad -\frac{1}{2} < x < \frac{1}{2}.$$

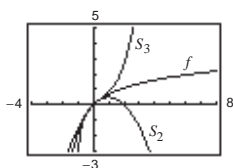
26. Since  $\int \frac{1}{4x^2+1} dx = \frac{1}{2} \arctan(2x)$ , we can use the result of Exercise 25 to obtain

$$\arctan(2x) = 2 \int \frac{1}{4x^2+1} dx = 2 \int \left[ \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n} \right] dx = C + 2 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1}, \quad -\frac{1}{2} < x \leq \frac{1}{2}.$$

To solve for  $C$ , let  $x = 0$  and conclude that  $C = 0$ . Therefore,

$$\arctan(2x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1}, \quad -\frac{1}{2} < x \leq \frac{1}{2}.$$

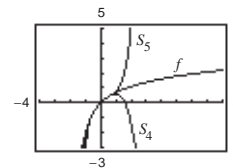
$$27. x - \frac{x^2}{2} \leq \ln(x+1) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$$



|                                     |       |       |       |       |       |       |
|-------------------------------------|-------|-------|-------|-------|-------|-------|
| $x$                                 | 0.0   | 0.2   | 0.4   | 0.6   | 0.8   | 1.0   |
| $x - \frac{x^2}{2}$                 | 0.000 | 0.180 | 0.320 | 0.420 | 0.480 | 0.500 |
| $\ln(x+1)$                          | 0.000 | 0.182 | 0.336 | 0.470 | 0.588 | 0.693 |
| $x - \frac{x^2}{2} + \frac{x^3}{3}$ | 0.000 | 0.183 | 0.341 | 0.492 | 0.651 | 0.833 |

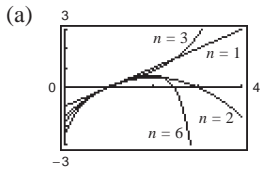
$$28. x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \leq \ln(x+1)$$

$$\leq x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$



|   |     |         |         |         |         |         |
|---|-----|---------|---------|---------|---------|---------|
| $x$   | 0.0 | 0.2     | 0.4     | 0.6     | 0.8     | 1.0     |
| $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$                 | 0.0 | 0.18227 | 0.33493 | 0.45960 | 0.54827 | 0.58333 |
| $\ln(x+1)$  | 0.0 | 0.18232 | 0.33647 | 0.47000 | 0.58779 | 0.69315 |
| $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$ | 0.0 | 0.18233 | 0.33698 | 0.47515 | 0.61380 | 0.78333 |

$$29. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n} = \frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots$$



(b) From Example 4,

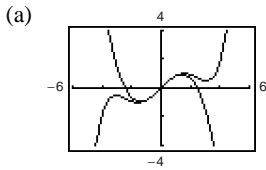
$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n} &= \sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^{n+1}}{n+1} \\ &= \ln x, \quad 0 < x \leq 2, \quad R = 1. \end{aligned}$$

(c)  $x = 0.5$ :

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1/2)^n}{n} = \sum_{n=1}^{\infty} \frac{-(1/2)^n}{n} \approx -0.693147$$

(d) This is an approximation of  $\ln(\frac{1}{2})$ . The error is approximately 0. [The error is less than the first omitted term,  $1/(51 \cdot 2^{51}) \approx 8.7 \times 10^{-18}$ .]

$$30. \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$



(b)  $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin x, \quad R = \infty$

(c)  $\sum_{n=0}^{\infty} \frac{(-1)^n (1/2)^{2n+1}}{(2n+1)!} \approx 0.4794255386$

(d) This is an approximation of  $\sin(\frac{1}{2})$ . The error is approximately 0.

31.  $g(x) = x$  line

Matches (c)

32.  $g(x) = x - \frac{x^3}{3}$ , cubic with 3 zeros.

Matches (d)

33.  $g(x) = x - \frac{x^3}{3} + \frac{x^5}{5}$

Matches (a)

34.  $g(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$

Matches (b)

**In Exercises 35-38,**  $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ .

35.  $\arctan \frac{1}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{(1/4)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)4^{2n+1}} = \frac{1}{4} - \frac{1}{192} + \frac{1}{5120} + \dots$

Since  $\frac{1}{5120} < 0.001$ , we can approximate the series by its first two terms:  $\arctan \frac{1}{4} \approx \frac{1}{4} - \frac{1}{192} \approx 0.245$ .

36.  $\arctan x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$

$$\int \arctan x^2 dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)} + C, \quad C = 0$$

$$\int_0^{3/4} \arctan x^2 dx = \sum_{n=0}^{\infty} (-1)^n \frac{(3/4)^{4n+3}}{(4n+3)(2n+1)} = \sum_{n=0}^{\infty} (-1)^n \frac{3^{4n+3}}{(4n+3)(2n+1)4^{4n+3}}$$

$$= \frac{27}{192} - \frac{2187}{344,064} + \frac{177,147}{230,686,720}$$

Since  $177,147/230,686,720 < 0.001$ , we can approximate the series by its first two terms: 0.134.

$$37. \quad \frac{\arctan x^2}{x} = \frac{1}{x} \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{2n+1}$$

$$\int \frac{\arctan x^2}{x} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(4n+2)(2n+1)} + C \quad (\text{Note: } C = 0)$$

$$\int_0^{1/2} \frac{\arctan x^2}{x} dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(4n+2)(2n+1)2^{4n+2}} = \frac{1}{8} - \frac{1}{1152} + \dots$$

Since  $\frac{1}{1152} < 0.001$ , we can approximate the series by its first term:  $\int_0^{1/2} \frac{\arctan x^2}{x} dx \approx 0.125$ .

$$38. \quad x^2 \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{2n+1}$$

$$\int x^2 \arctan x dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{(2n+4)(2n+1)}$$

$$\int_0^{1/2} x^2 \arctan x dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+4)(2n+1)2^{2n+4}} = \frac{1}{64} - \frac{1}{1152} + \dots$$

Since  $\frac{1}{1152} < 0.001$ , we can approximate the series by its first term:  $\int_0^{1/2} x^2 \arctan x dx \approx 0.016$ .

In Exercises 39–43, use  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ ,  $|x| < 1$ .

$$39. \quad \frac{1}{(1-x)^2} = \frac{d}{dx} \left[ \frac{1}{1-x} \right] = \frac{d}{dx} \left[ \sum_{n=0}^{\infty} x^n \right] = \sum_{n=1}^{\infty} nx^{n-1}, \quad |x| < 1$$

$$40. \quad \frac{x}{(1-x)^2} = x \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^n, \quad |x| < 1$$

$$\begin{aligned} 41. \quad \frac{1+x}{(1-x)^2} &= \frac{1}{(1-x)^2} + \frac{x}{(1-x)^2} \\ &= \sum_{n=1}^{\infty} n(x^{n-1} + x^n), \quad |x| < 1 \\ &= \sum_{n=0}^{\infty} (2n+1)x^n, \quad |x| < 1 \end{aligned}$$

$$42. \quad \frac{x(1+x)}{(1-x)^2} = x \sum_{n=0}^{\infty} (2n+1)x^n = \sum_{n=0}^{\infty} (2n+1)x^{n+1}, \quad |x| < 1$$

(See Exercise 41.)

$$43. \quad P(n) = \left(\frac{1}{2}\right)^n$$

$$\begin{aligned} E(n) &= \sum_{n=1}^{\infty} nP(n) = \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n = \frac{1}{2} \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^{n-1} \\ &= \frac{1}{2} \frac{1}{[1 - (1/2)]^2} = 2 \end{aligned}$$

Since the probability of obtaining a head on a single toss is  $\frac{1}{2}$ , it is expected that, on average, a head will be obtained in two tosses.

In Exercise 44,  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ ,  $|x| < 1$ .

$$44. \quad (a) \quad \frac{1}{3} \sum_{n=1}^{\infty} n \left(\frac{2}{3}\right)^n = \frac{2}{9} \sum_{n=1}^{\infty} n \left(\frac{2}{3}\right)^{n-1} = \frac{2}{9} \frac{1}{[1 - (2/3)]^2} = 2$$

$$(b) \quad \frac{1}{10} \sum_{n=1}^{\infty} n \left(\frac{9}{10}\right)^n = \frac{9}{100} \sum_{n=1}^{\infty} n \left(\frac{9}{10}\right)^{n-1} = \frac{9}{100} \cdot \frac{1}{[1 - (9/10)]^2} = 9$$



45. Replace  $x$  with  $(-x)$ .      46. Replace  $x$  with  $x^2$ .      47. Replace  $x$  with  $(-x)$  and multiply the series by 5.      48. Integrate the series and multiply by  $(-1)$ .

49. Let  $\arctan x + \arctan y = \theta$ . Then,

$$\begin{aligned}\tan(\arctan x + \arctan y) &= \tan \theta \\ \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} &= \tan \theta \\ \frac{x + y}{1 - xy} &= \tan \theta\end{aligned}$$

$$\arctan\left(\frac{x + y}{1 - xy}\right) = \theta. \text{ Therefore, } \arctan x + \arctan y = \arctan\left(\frac{x + y}{1 - xy}\right) \text{ for } xy \neq 1.$$

50. (a) From Exercise 49, we have

$$\begin{aligned}\arctan \frac{120}{119} - \arctan \frac{1}{239} &= \arctan \frac{120}{119} + \arctan\left(-\frac{1}{239}\right) \\ &= \arctan\left[\frac{(120/119) + (-1/239)}{1 - (120/119)(-1/239)}\right] = \arctan\left(\frac{28,561}{28,561}\right) = \arctan 1 = \frac{\pi}{4}\end{aligned}$$

$$(b) \quad 2 \arctan \frac{1}{5} = \arctan \frac{1}{5} + \arctan \frac{1}{5} = \arctan\left[\frac{2(1/5)}{1 - (1/5)^2}\right] = \arctan \frac{10}{24} = \arctan \frac{5}{12}$$

$$4 \arctan \frac{1}{5} = 2 \arctan \frac{1}{5} + 2 \arctan \frac{1}{5} = \arctan \frac{5}{12} + \arctan \frac{5}{12} = \arctan\left[\frac{2(5/12)}{1 - (5/12)^2}\right] = \arctan \frac{120}{119}$$

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \arctan \frac{120}{119} - \arctan \frac{1}{239} = \frac{\pi}{4} \text{ (see part (a).)}$$

$$51. (a) \quad 2 \arctan \frac{1}{2} = \arctan \frac{1}{2} + \arctan \frac{1}{2} = \arctan\left[\frac{\frac{1}{2} + \frac{1}{2}}{1 - (1/2)^2}\right] = \arctan \frac{4}{3}$$

$$2 \arctan \frac{1}{2} - \arctan \frac{1}{7} = \arctan \frac{4}{3} + \arctan\left(-\frac{1}{7}\right) = \arctan\left[\frac{(4/3) - (1/7)}{1 + (4/3)(1/7)}\right] = \arctan \frac{25}{25} = \arctan 1 = \frac{\pi}{4}$$

$$(b) \quad \pi = 8 \arctan \frac{1}{2} - 4 \arctan \frac{1}{7} \approx 8\left[\frac{1}{2} - \frac{(0.5)^3}{3} + \frac{(0.5)^5}{5} - \frac{(0.5)^7}{7}\right] - 4\left[\frac{1}{7} - \frac{(1/7)^3}{3} + \frac{(1/7)^5}{5} - \frac{(1/7)^7}{7}\right] \approx 3.14$$

$$52. (a) \quad \arctan \frac{1}{2} + \arctan \frac{1}{3} = \arctan\left[\frac{(1/2) + (1/3)}{1 - (1/2)(1/3)}\right] = \arctan\left(\frac{5/6}{5/6}\right) = \frac{\pi}{4}$$

$$\begin{aligned}(b) \quad \pi &= 4\left[\arctan \frac{1}{2} + \arctan \frac{1}{3}\right] \\ &= 4\left[\frac{1}{2} - \frac{(1/2)^3}{3} + \frac{(1/2)^5}{5} - \frac{(1/2)^7}{7}\right] + 4\left[\frac{1}{3} - \frac{(1/3)^3}{3} + \frac{(1/3)^5}{5} - \frac{(1/3)^7}{7}\right] \approx 4(0.4635) + 4(0.3217) \approx 3.14\end{aligned}$$

53. From Exercise 21, we have

$$\begin{aligned}\ln(x + 1) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.\end{aligned}$$

$$\begin{aligned}\text{Thus, } \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2^n n} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (1/2)^n}{n} \\ &= \ln\left(\frac{1}{2} + 1\right) = \ln \frac{3}{2} \approx 0.4055.\end{aligned}$$

54. From Exercise 53, we have

$$\begin{aligned}\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{3^n n} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (1/3)^n}{n} \\ &= \ln\left(\frac{1}{3} + 1\right) = \ln \frac{4}{3} \approx 0.2877.\end{aligned}$$

55. From Exercise 53, we have

$$\begin{aligned}\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{5^n n} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2/5)^n}{n} \\ &= \ln\left(\frac{2}{5} + 1\right) = \ln\frac{7}{5} \approx 0.3365.\end{aligned}$$

57. From Exercise 56, we have

$$\begin{aligned}\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n+1}(2n+1)} &= \sum_{n=0}^{\infty} (-1)^n \frac{(1/2)^{2n+1}}{2n+1} \\ &= \arctan\frac{1}{2} \approx 0.4636.\end{aligned}$$

59.  $f(x) = \arctan x$  is an odd function (symmetric to the origin).

61. The series in Exercise 56 converges to its sum at a slower rate because its terms approach 0 at a much slower rate.

63. Because the first series is the derivative of the second series, the second series converges for  $|x + 1| < 4$  (and perhaps at the endpoints,  $x = 3$  and  $x = -5$ .)

$$65. \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)}$$

From Example 5 we have  $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ .

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(\sqrt{3})^{2n}(2n+1)} \frac{\sqrt{3}}{\sqrt{3}} \\ &= \sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n (1/\sqrt{3})^{2n+1}}{2n+1} \\ &= \sqrt{3} \arctan\left(\frac{1}{\sqrt{3}}\right) \\ &= \sqrt{3} \left(\frac{\pi}{6}\right) \approx 0.9068997\end{aligned}$$

56. From Example 5, we have  $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ .

$$\begin{aligned}\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} &= \sum_{n=0}^{\infty} (-1)^n \frac{(1)^{2n+1}}{2n+1} \\ &= \arctan 1 = \frac{\pi}{4} \approx 0.7854\end{aligned}$$

58. From Exercise 56, we have

$$\begin{aligned}\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{3^{2n-1}(2n-1)} &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(1/3)^{2n+1}}{2n+1} \\ &= \arctan\frac{1}{3} \approx 0.3218.\end{aligned}$$

60. The approximations of degree 3, 7, 11, . . . ,  $(4n - 1, n = 1, 2, \dots)$  have relative extrema.

62. Because  $\frac{d}{dx} \left[ \sum_{n=0}^{\infty} a_n x^n \right] = \sum_{n=1}^{\infty} n a_n x^{n-1}$ , the radius of convergence is the same, 3.

64. Using a graphing utility, you obtain the following partial sums for the left hand side. Note that  $1/\pi \approx 0.3183098862$ .

$$n = 0: S_0 \approx 0.3183098784$$

$$n = 1: S_1 \approx 0.3183098862$$

$$\begin{aligned}66. \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{3^{2n+1}(2n+1)!} &= \sum_{n=0}^{\infty} (-1)^n \frac{(\pi/3)^{2n+1}}{(2n+1)!} \\ &= \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \approx 0.866025\end{aligned}$$

## Section 9.10 Taylor and Maclaurin Series

1. For  $c = 0$ , we have:

$$f(x) = e^{2x}$$

$$f^{(n)}(x) = 2^n e^{2x} \Rightarrow f^{(n)}(0) = 2^n$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}.$$

2. For  $c = 0$ , we have:

$$\begin{aligned} f(x) &= e^{3x} \\ f^{(n)}(x) &= 3^n e^{3x} \implies f^{(n)}(0) = 3^n \\ e^{3x} &= 1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}. \end{aligned}$$

3. For  $c = \pi/4$ , we have:

$$\begin{aligned} f(x) &= \cos(x) & f\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} \\ f'(x) &= -\sin(x) & f'\left(\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2} \\ f''(x) &= -\cos(x) & f''\left(\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2} \\ f'''(x) &= \sin(x) & f'''\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} \\ f^{(4)}(x) &= \cos(x) & f^{(4)}\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} \end{aligned}$$

and so on. Therefore, we have:

$$\begin{aligned} \cos x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/4)[x - (\pi/4)]^n}{n!} \\ &= \frac{\sqrt{2}}{2} \left[ 1 - \left(x - \frac{\pi}{4}\right) - \frac{[x - (\pi/4)]^2}{2!} + \frac{[x - (\pi/4)]^3}{3!} + \frac{[x - (\pi/4)]^4}{4!} - \dots \right] \\ &= \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2} [x - (\pi/4)]^n}{n!}. \end{aligned}$$

[Note:  $(-1)^{n(n+1)/2} = 1, -1, -1, 1, 1, -1, -1, 1, \dots$ ]

4. For  $c = \pi/4$ , we have:

$$\begin{aligned} f(x) &= \sin x & f\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} \\ f'(x) &= \cos x & f'\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} \\ f''(x) &= -\sin x & f''\left(\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2} \\ f'''(x) &= -\cos x & f'''\left(\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2} \\ f^{(4)}(x) &= \sin x & f^{(4)}\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} \end{aligned}$$

and so on. Therefore we have:

$$\begin{aligned} \sin x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/4)[x - (\pi/4)]^n}{n!} \\ &= \frac{\sqrt{2}}{2} \left[ 1 + \left(x - \frac{\pi}{4}\right) - \frac{[x - (\pi/4)]^2}{2!} - \frac{[x - (\pi/4)]^3}{3!} + \frac{[x - (\pi/4)]^4}{4!} + \dots \right] \\ &= \frac{\sqrt{2}}{2} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2} [x - (\pi/4)]^{n+1}}{(n+1)!} + 1 \right\}. \end{aligned}$$

5. For  $c = 1$ , we have,

$$f(x) = \ln x \quad f(1) = 0$$

$$f'(x) = \frac{1}{x} \quad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \quad f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \quad f'''(1) = 2$$

$$f^{(4)}(x) = -\frac{6}{x^4} \quad f^{(4)}(1) = -6$$

$$f^{(5)}(x) = \frac{24}{x^5} \quad f^{(5)}(1) = 24$$

and so on. Therefore, we have:

$$\begin{aligned} \ln x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} \\ &= 0 + (x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!} + \frac{24(x-1)^5}{5!} - \dots \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}. \end{aligned}$$

6. For  $c = 1$ , we have:

$$f(x) = e^x$$

$$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(1) = e$$

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} = e \left[ 1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} + \dots \right] = e \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}.$$

7. For  $c = 0$ , we have:

$$f(x) = \sin 2x \quad f(0) = 0$$

$$f'(x) = 2 \cos 2x \quad f'(0) = 2$$

$$f''(x) = -4 \sin 2x \quad f''(0) = 0$$

$$f'''(x) = -8 \cos 2x \quad f'''(0) = -8$$

$$f^{(4)}(x) = 16 \sin 2x \quad f^{(4)}(0) = 0$$

$$f^{(5)}(x) = 32 \cos 2x \quad f^{(5)}(0) = 32$$

$$f^{(6)}(x) = -64 \sin 2x \quad f^{(6)}(0) = 0$$

$$f^{(7)}(x) = -128 \cos 2x \quad f^{(7)}(0) = -128$$

and so on. Therefore, we have:

$$\begin{aligned} \sin 2x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0 + 2x + \frac{0x^2}{2!} - \frac{8x^3}{3!} + \frac{0x^4}{4!} + \frac{32x^5}{5!} + \frac{0x^6}{6!} - \frac{128x^7}{7!} + \dots \\ &= 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}. \end{aligned}$$

8. For  $c = 0$ , we have:

$$\begin{aligned} f(x) &= \ln(x^2 + 1) & f(0) &= 0 \\ f'(x) &= \frac{2x}{x^2 + 1} & f'(0) &= 0 \\ f''(x) &= \frac{2 - 2x^2}{(x^2 + 1)^2} & f''(0) &= 2 \\ f'''(x) &= \frac{4x(x^2 - 3)}{(x^2 + 1)^3} & f'''(0) &= 0 \\ f^{(4)}(x) &= \frac{12(-x^4 + 6x^2 - 1)}{(x^2 + 1)^4} & f^{(4)}(0) &= -12 \\ f^{(5)}(x) &= \frac{48x(x^4 - 10x^2 + 5)}{(x^2 + 1)^5} & f^{(5)}(0) &= 0 \\ f^{(6)}(x) &= \frac{-240(5x^6 - 15x^4 + 15x^2 - 1)}{(x^2 + 1)^6} & f^{(6)}(0) &= 240 \end{aligned}$$

and so on. Therefore, we have:

$$\begin{aligned} \ln(x^2 + 1) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0 + 0x + \frac{2x^2}{2!} + \frac{0x^3}{3!} - \frac{12x^4}{4!} + \frac{0x^5}{5!} + \frac{240x^6}{6!} + \cdots \\ &= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}. \end{aligned}$$

9. For  $c = 0$ , we have:

$$\begin{aligned} f(x) &= \sec(x) & f(0) &= 1 \\ f'(x) &= \sec(x) \tan(x) & f'(0) &= 0 \\ f''(x) &= \sec^3(x) + \sec(x) \tan^2(x) & f''(0) &= 1 \\ f'''(x) &= 5 \sec^3(x) \tan(x) + \sec(x) \tan^3(x) & f'''(0) &= 0 \\ f^{(4)}(x) &= 5 \sec^5(x) + 18 \sec^3(x) \tan^2(x) + \sec(x) \tan^4(x) & f^{(4)}(0) &= 5 \\ \sec(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \cdots \end{aligned}$$

10. For  $c = 0$ , we have:

$$\begin{aligned} f(x) &= \tan(x) & f(0) &= 0 \\ f'(x) &= \sec^2(x) & f'(0) &= 1 \\ f''(x) &= 2 \sec^2(x) \tan(x) & f''(0) &= 0 \\ f'''(x) &= 2[\sec^4(x) + 2 \sec^2(x) \tan^2(x)] & f'''(0) &= 2 \\ f^{(4)}(x) &= 8[\sec^4(x) \tan(x) + \sec^2(x) \tan^3(x)] & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= 8[2 \sec^6(x) + 11 \sec^4(x) \tan^2(x) + 2 \sec^2(x) \tan^4(x)] & f^{(5)}(0) &= 16 \\ \tan(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \cdots = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots \end{aligned}$$

11. The Maclaurin series for  $f(x) = \cos x$  is  $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ .

Because  $f^{(n+1)}(x) = \pm \sin x$  or  $\pm \cos x$ , we have  $|f^{(n+1)}(z)| \leq 1$  for all  $z$ . Hence by Taylor's Theorem,

$$0 \leq |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

Since  $\lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ , it follows that  $R_n(x) \rightarrow 0$  as  $n \rightarrow \infty$ . Hence, the Maclaurin series for  $\cos x$  converges to  $\cos x$  for all  $x$ .

12. The Maclaurin series for  $f(x) = e^{-2x}$  is  $\sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}$ .

$f^{(n+1)}(x) = (-2)^{n+1} e^{-2x}$ . Hence, by Taylor's Theorem,

$$0 \leq |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| = \left| \frac{(-2)^{n+1} e^{-2z}}{(n+1)!} x^{n+1} \right|.$$

Since  $\lim_{n \rightarrow \infty} \left| \frac{(-2)^{n+1} x^{n+1}}{(n+1)!} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x)^{n+1}}{(n+1)!} \right| = 0$ , it follows that  $R_n(x) \rightarrow 0$  as  $n \rightarrow \infty$ .

Hence, the Maclaurin Series for  $e^{-2x}$  converges to  $e^{-2x}$  for all  $x$ .

13. The Maclaurin series for  $f(x) = \sinh x$  is  $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ .

$f^{(n+1)}(x) = \sinh x$  (or  $\cosh x$ ). For fixed  $x$ ,

$$0 \leq |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| = \left| \frac{\sinh(z)}{(n+1)!} x^{n+1} \right| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(The argument is the same if  $f^{(n+1)}(x) = \cosh x$ ). Hence, the Maclaurin series for  $\sinh x$  converges to  $\sinh x$  for all  $x$ .

14. The Maclaurin series for  $f(x) = \cosh x$  is  $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ .

$f^{(n+1)}(x) = \sinh x$  (or  $\cosh x$ ). For fixed  $x$ ,

$$0 \leq |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| = \left| \frac{\sinh(z)}{(n+1)!} x^{n+1} \right| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(The argument is the same if  $f^{(n+1)}(x) = \cosh x$ ). Hence, the Maclaurin series for  $\cosh x$  converges to  $\cosh x$  for all  $x$ .

15. Since  $(1+x)^{-k} = 1 - kx + \frac{k(k+1)x^2}{2!} - \frac{k(k+1)(k+2)x^3}{3!} + \dots$ , we have

$$\begin{aligned} (1+x)^{-2} &= 1 - 2x + \frac{2(3)x^2}{2!} - \frac{2(3)(4)x^3}{3!} + \frac{2(3)(4)(5)x^4}{4!} - \dots = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \dots \\ &= \sum_{n=0}^{\infty} (-1)^n (n+1)x^n. \end{aligned}$$

16. Since  $(1+x)^{-k} = 1 - kx + \frac{k(k+1)x^2}{2!} - \frac{k(k+1)(k+2)x^3}{3!} + \dots$ , we have

$$\begin{aligned} \left[ 1 + (-x) \right]^{-1/2} &= 1 + \left( \frac{1}{2} \right) x + \frac{(1/2)(3/2)x^2}{2!} + \frac{(1/2)(3/2)(5/2)x^3}{3!} + \dots \\ &= 1 + \frac{x}{2} + \frac{(1)(3)x^2}{2^2 2!} + \frac{(1)(3)(5)x^3}{2^3 3!} + \dots \\ &= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)x^n}{2^n n!}. \end{aligned}$$

17.  $\frac{1}{\sqrt{4+x^2}} = \left(\frac{1}{2}\right) \left[1 + \left(\frac{x}{2}\right)^2\right]^{-1/2}$  and since  $(1+x)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}$ , we have

$$\frac{1}{\sqrt{4+x^2}} = \frac{1}{2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)(x/2)^{2n}}{2^n n!}\right] = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^{3n+1}n!}.$$

18.  $(1+x)^{1/4} = 1 + \frac{1}{4}x + \frac{(1/4)(-3/4)}{2!}x^2 + \frac{(1/4)(-3/4)(-7/4)}{3!}x^3 + \cdots$

$$= 1 + \frac{1}{4}x - \frac{3}{4^2 2!}x^2 + \frac{3 \cdot 7}{4^3 3!}x^3 - \frac{3 \cdot 7 \cdot 11}{4^4 4!}x^4 + \cdots$$

$$= 1 + \frac{1}{4}x + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 3 \cdot 7 \cdot 11 \cdots (4n-5)}{4^n n!} x^n$$

19. Since  $(1+x)^{1/2} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)x^n}{2^n n!}$

we have  $(1+x^2)^{1/2} = 1 + \frac{x^2}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)x^{2n}}{2^n n!}$ .

20. Since  $(1+x)^{1/2} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)x^n}{2^n n!}$

we have  $(1+x^3)^{1/2} = 1 + \frac{x^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)x^{3n}}{2^n n!}$ .

21.  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

$$e^{x^2/2} = \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = 1 + \frac{x^2}{2} + \frac{x^4}{2^2 2!} + \frac{x^6}{2^3 3!} + \frac{x^8}{2^4 4!} + \cdots$$

22.  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

$$e^{-3x} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^n}{n!} = 1 - 3x + \frac{9x^2}{2!} - \frac{27x^3}{3!} + \frac{81x^4}{4!} - \frac{243x^5}{5!} + \cdots$$

23.  $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\sin 3x = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!}$$

24.  $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$\cos 4x = \sum_{n=0}^{\infty} \frac{(-1)^n (4x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n}}{(2n)!}$$

$$= 1 - \frac{16x^2}{2!} + \frac{256x^4}{4!} - \cdots$$

25.  $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$$\cos x^{3/2} = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{3/2})^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{(2n)!}$$

$$= 1 - \frac{x^3}{2!} + \frac{x^6}{4!} - \cdots$$

26.  $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$2 \sin x^3 = 2 \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!}$$

$$= 2 \left[ x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots \right]$$

$$= 2x^3 - \frac{2x^9}{3!} + \frac{2x^{15}}{5!} - \cdots$$

$$27. \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$$

$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \frac{2x^7}{7!} + \cdots$$

$$\begin{aligned} \sinh(x) &= \frac{1}{2}(e^x - e^{-x}) \\ &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

$$28. \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \cdots$$

$$2 \cosh(x) = e^x + e^{-x} = \sum_{n=0}^{\infty} 2 \frac{x^{2n}}{(2n)!}$$

$$29. \quad \cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$$

$$= \frac{1}{2} \left[ 1 + 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots \right] = \frac{1}{2} \left[ 1 + \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} \right]$$

30. The formula for the binomial series gives  $(1+x)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}$ , which implies that

$$(1+x^2)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^n n!}$$

$$\ln(x + \sqrt{x^2 + 1}) = \int \frac{1}{\sqrt{x^2 + 1}} dx$$

$$= x + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n+1}}{2^n (2n+1)n!}$$

$$= x - \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} - \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots$$

$$31. \quad x \sin x = x \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right)$$

$$= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}$$

$$32. \quad x \cos x = x \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right)$$

$$= x - \frac{x^3}{2!} + \frac{x^5}{4!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}$$

$$33. \quad \frac{\sin x}{x} = \frac{x - (x^3/3!) + (x^5/5!) - \cdots}{x}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}, x \neq 0$$

$$34. \quad \frac{\arcsin x}{x} = \sum_{n=0}^{\infty} \frac{(2n)! x^{2n+1}}{(2^n n!)^2 (2n+1)} \cdot \frac{1}{x}$$

$$= \sum_{n=0}^{\infty} \frac{(2n)! x^{2n}}{(2^n n!)^2 (2n+1)}, x \neq 0$$

$$35. \quad e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \cdots$$

$$e^{-ix} = 1 - ix + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \frac{(-ix)^4}{4!} + \cdots = 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \cdots$$

$$e^{ix} - e^{-ix} = 2ix - \frac{2ix^3}{3!} + \frac{2ix^5}{5!} - \frac{2ix^7}{7!} + \cdots$$

$$\frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin(x)$$

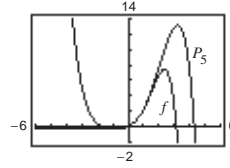


$$36. e^{ix} + e^{-ix} = 2 - \frac{2x^2}{2!} + \frac{2x^4}{4!} - \frac{2x^6}{6!} + \cdots \quad (\text{See Exercise 35.})$$

$$\frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos(x)$$

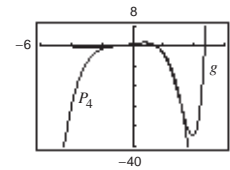
$$37. f(x) = e^x \sin x$$

$$\begin{aligned} &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) \\ &= x + x^2 + \left(\frac{x^3}{2} - \frac{x^3}{6}\right) + \left(\frac{x^4}{6} - \frac{x^4}{6}\right) + \left(\frac{x^5}{120} - \frac{x^5}{12} + \frac{x^5}{24}\right) + \cdots \\ &= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots \end{aligned}$$



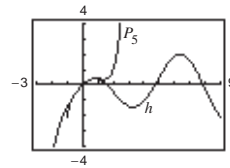
$$38. g(x) = e^x \cos x$$

$$\begin{aligned} &= \left(1 + x + \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^4}{24} + \cdots\right) \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) \\ &= 1 + x + \left(\frac{x^2}{2} - \frac{x^2}{2}\right) + \left(\frac{x^3}{6} - \frac{x^3}{2}\right) + \left(\frac{x^4}{24} - \frac{x^4}{4} + \frac{x^4}{24}\right) + \cdots \\ &= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots \end{aligned}$$



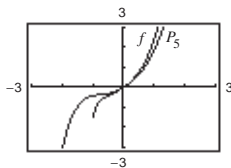
$$39. h(x) = \cos x \ln(1+x)$$

$$\begin{aligned} &= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right) \\ &= x - \frac{x^2}{2} + \left(\frac{x^3}{3} - \frac{x^3}{2}\right) + \left(\frac{x^4}{4} - \frac{x^4}{4}\right) + \left(\frac{x^5}{5} - \frac{x^5}{6} + \frac{x^5}{24}\right) + \cdots \\ &= x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{3x^5}{40} + \cdots \end{aligned}$$



$$40. f(x) = e^x \ln(1+x)$$

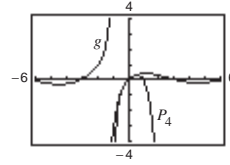
$$\begin{aligned} &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right) \\ &= x + \left(x^2 - \frac{x^2}{2}\right) + \left(\frac{x^3}{3} - \frac{x^3}{2} + \frac{x^3}{2}\right) + \left(-\frac{x^4}{4} + \frac{x^4}{3} - \frac{x^4}{4} + \frac{x^4}{6}\right) + \left(\frac{x^5}{5} - \frac{x^5}{4} + \frac{x^5}{6} - \frac{x^5}{12} + \frac{x^5}{24}\right) + \cdots \\ &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40} + \cdots \end{aligned}$$



41.  $g(x) = \frac{\sin x}{1+x}$ . Divide the series for  $\sin x$  by  $(1+x)$ .

$$\begin{array}{r}
 x - x^2 + \frac{5x^3}{6} - \frac{5x^4}{6} + \dots \\
 1+x \overline{) x + 0x^2 - \frac{x^3}{6} + 0x^4 + \frac{x^5}{120} + \dots} \\
 \underline{x + x^2} \phantom{+ \dots} \\
 -x^2 - \frac{x^3}{6} \phantom{+ \dots} \\
 \underline{-x^2 - \frac{x^3}{6}} \phantom{+ \dots} \\
 \frac{5x^3}{6} + 0x^4 \phantom{+ \dots} \\
 \underline{\frac{5x^3}{6} + \frac{5x^4}{6}} \phantom{+ \dots} \\
 -\frac{5x^4}{6} + \frac{x^5}{120} \phantom{+ \dots} \\
 \underline{-\frac{5x^4}{6} - \frac{5x^5}{6}} \phantom{+ \dots} \\
 \vdots
 \end{array}$$

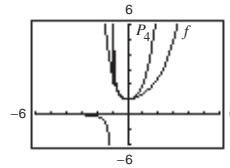
$$g(x) = x - x^2 + \frac{5x^3}{6} - \frac{5x^4}{6} + \dots$$



42.  $f(x) = \frac{e^x}{1+x}$ . Divide the series for  $e^x$  by  $(1+x)$ .

$$\begin{array}{r}
 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} + \dots \\
 1+x \overline{) 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots} \\
 \underline{1 + x} \phantom{+ \dots} \\
 0 + \frac{x^2}{2} + \frac{x^3}{6} \phantom{+ \dots} \\
 \underline{\frac{x^2}{2} + \frac{x^3}{2}} \phantom{+ \dots} \\
 -\frac{x^3}{3} + \frac{x^4}{24} \phantom{+ \dots} \\
 \underline{-\frac{x^3}{3} - \frac{x^4}{3}} \phantom{+ \dots} \\
 \frac{3x^4}{8} + \frac{x^5}{120} \phantom{+ \dots} \\
 \underline{\frac{3x^4}{8} + \frac{3x^5}{8}} \phantom{+ \dots} \\
 \vdots
 \end{array}$$

$$f(x) = 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} - \dots$$



43.  $y = x^2 - \frac{x^4}{3!} = x\left(x - \frac{x^3}{3!}\right)$

$$f(x) = x \sin x$$

Matches (c)

44.  $y = x - \frac{x^3}{2!} + \frac{x^5}{4!} = x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)$

$$f(x) = x \cos x$$

Matches (d)

45.  $y = x + x^2 + \frac{x^3}{2!} = x\left(1 + x + \frac{x^2}{2!}\right)$

$$f(x) = xe^x$$

Matches (a)

46.  $y = x^2 - x^3 + x^4 = x^2(1 - x + x^2)$

$$f(x) = x^2 \frac{1}{1+x}$$

Matches (b)

$$\begin{aligned}
 47. \int_0^x (e^{-t^2} - 1) dt &= \int_0^x \left[ \left( \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} \right) - 1 \right] dt \\
 &= \int_0^x \left[ \sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+2}}{(n+1)!} \right] dt = \left[ \sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+3}}{(2n+3)(n+1)!} \right]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)(n+1)!}
 \end{aligned}$$

$$\begin{aligned}
 48. \int_0^x \sqrt{1+t^3} dt &= \int_0^x \left[ 1 + \frac{t^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3) t^{3n}}{2^n n!} \right] dt \\
 &= \left[ t + \frac{t^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3) t^{3n+1}}{(3n+1) 2^n n!} \right]_0^x \\
 &= x + \frac{x^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3) x^{3n+1}}{(3n+1) 2^n n!}
 \end{aligned}$$

$$49. \text{ Since } \ln x = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots, \quad (0 < x \leq 2)$$

$$\text{we have } \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \approx 0.6931. \quad (10,001 \text{ terms})$$

$$50. \text{ Since } \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \text{ we have}$$

$$\sin(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots \approx 0.8415. \quad (4 \text{ terms})$$

$$51. \text{ Since } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

$$\text{we have } e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{2^n}{n!} \approx 7.3891. \quad (12 \text{ terms})$$

$$52. \text{ Since } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots, \text{ we have } e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$$

$$\text{and } \frac{e-1}{e} = 1 - e^{-1} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{7!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \approx 0.6321. \quad (6 \text{ terms})$$

53. Since

$$\begin{aligned}
 \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \\
 1 - \cos x &= \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!} \\
 \frac{1 - \cos x}{x} &= \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+2)!}
 \end{aligned}$$

$$\text{we have } \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = \lim_{x \rightarrow 0} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+2)!} = 0.$$

54. Since

$$\begin{aligned}
 \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
 \frac{\sin x}{x} &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}
 \end{aligned}$$

$$\text{we have } \lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1.$$

$$55. \int_0^1 \frac{\sin x}{x} dx = \int_0^1 \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} \right] dx = \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} \right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}$$

Since  $1/(7 \cdot 7!) < 0.0001$ , we need three terms:

$$\int_0^1 \frac{\sin x}{x} dx = 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} - \cdots \approx 0.9461. \quad (\text{using three nonzero terms})$$

**Note:** We are using  $\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$ .

$$56. \int_0^{1/2} \frac{\arctan x}{x} dx = \int_0^{1/2} \left( 1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \cdots \right) dx = \left[ x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \cdots \right]_0^{1/2}$$

Since  $1/(9 \cdot 2^{29}) < 0.0001$ , we have

$$\int_0^{1/2} \frac{\arctan x}{x} dx \approx \left( \frac{1}{2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} + \frac{1}{9 \cdot 2^9} \right) \approx 0.4872.$$

**Note:** We are using  $\lim_{x \rightarrow 0^+} \frac{\arctan x}{x} = 1$ .

$$57. \int_{0.1}^{0.3} \sqrt{1+x^3} dx = \int_{0.1}^{0.3} \left( 1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16} - \frac{5x^{12}}{128} + \cdots \right) dx = \left[ x + \frac{x^4}{8} - \frac{x^7}{56} + \frac{x^{10}}{160} - \frac{5x^{13}}{1664} + \cdots \right]_{0.1}^{0.3}$$

Since  $\frac{1}{56}(0.3^7 - 0.1^7) < 0.0001$ , we need two terms.

$$\int_{0.1}^{0.3} \sqrt{1+x^3} dx = \left[ (0.3 - 0.1) + \frac{1}{8}(0.3^4 - 0.1^4) \right] \approx 0.201.$$

$$58. \int_0^{1/4} x \ln(x+1) dx = \int_0^{1/4} \left( x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots \right) dx = \left[ \frac{x^3}{3} - \frac{x^4}{4 \cdot 2} + \frac{x^5}{5 \cdot 3} - \frac{x^6}{6 \cdot 4} + \cdots \right]_0^{1/4}$$

Since  $\frac{(1/4)^5}{15} < 0.0001$ ,  $\int_0^{1/4} x \ln(x+1) dx \approx \frac{(1/4)^3}{3} - \frac{(1/4)^4}{8} \approx 0.00472$ .

$$59. \int_0^{\pi/2} \sqrt{x} \cos x dx = \int_0^{\pi/2} \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{(4n+1)/2}}{(2n)!} \right] dx = \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{(4n+3)/2}}{\left(\frac{4n+3}{2}\right)(2n)!} \right]_0^{\pi/2} = \left[ \sum_{n=0}^{\infty} \frac{(-1)^n 2x^{(4n+3)/2}}{(4n+3)(2n)!} \right]_0^{\pi/2}$$

Since  $2(\pi/2)^{23/2}/(23 \cdot 10!) < 0.0001$ , we need five terms.

$$\int_0^1 \sqrt{x} \cos x dx = 2 \left[ \frac{(\pi/2)^{3/2}}{3} - \frac{(\pi/2)^{7/2}}{14} + \frac{(\pi/2)^{11/2}}{264} - \frac{(\pi/2)^{15/2}}{10,800} + \frac{(\pi/2)^{19/2}}{766,080} \right] \approx 0.7040.$$

$$60. \int_{0.5}^1 \cos \sqrt{x} dx = \int_{0.5}^1 \left( 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots \right) dx = \left[ x - \frac{x^2}{2(2!)} + \frac{x^3}{3(4!)} - \frac{x^4}{4(6!)} + \frac{x^5}{5(8!)} - \cdots \right]_{0.5}^1$$

Since  $\frac{1}{201,600}(1 - 0.5^5) < 0.0001$ , we have

$$\int_{0.5}^1 \cos \sqrt{x} dx \approx \left[ (1 - 0.5) - \frac{1}{4}(1 - 0.5^2) + \frac{1}{72}(1 - 0.5^3) - \frac{1}{2880}(1 - 0.5^4) + \frac{1}{201,600}(1 - 0.5^5) \right] \approx 0.3243.$$

61. From Exercise 21, we have

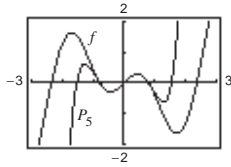
$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-x^2/2} dx &= \frac{1}{\sqrt{2\pi}} \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!} dx = \frac{1}{\sqrt{2\pi}} \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n! (2n+1)} \right]_0^1 = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n! (2n+1)} \\ &\approx \frac{1}{\sqrt{2\pi}} \left[ 1 - \frac{1}{2 \cdot 1 \cdot 3} + \frac{1}{2^2 \cdot 2! \cdot 5} - \frac{1}{2^3 \cdot 3! \cdot 7} \right] \approx 0.3414. \end{aligned}$$

62. From Exercise 21, we have

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_1^2 e^{-x^2/2} dx &= \frac{1}{\sqrt{2\pi}} \int_1^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!} dx = \frac{1}{\sqrt{2\pi}} \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n! (2n+1)} \right]_1^2 \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n (2^{2n+1} - 1)}{2^n n! (2n+1)} \\ &\approx \frac{1}{\sqrt{2\pi}} \left[ 1 - \frac{7}{2 \cdot 1 \cdot 3} + \frac{31}{2^2 \cdot 2! \cdot 5} - \frac{127}{2^3 \cdot 3! \cdot 7} + \frac{511}{2^4 \cdot 4! \cdot 9} - \frac{2047}{2^5 \cdot 5! \cdot 11} \right. \\ &\quad \left. + \frac{8191}{2^6 \cdot 6! \cdot 13} - \frac{32,767}{2^7 \cdot 7! \cdot 15} + \frac{131,071}{2^8 \cdot 8! \cdot 17} - \frac{524,287}{2^9 \cdot 9! \cdot 19} \right] \approx 0.1359. \end{aligned}$$

63.  $f(x) = x \cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{(2n)!}$

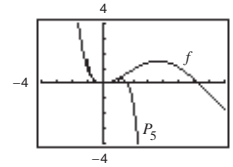
$$P_5(x) = x - 2x^3 + \frac{2x^5}{3}$$



The polynomial is a reasonable approximation on the interval  $[-\frac{3}{4}, \frac{3}{4}]$ .

64.  $f(x) = \sin \frac{x}{2} \ln(1+x)$

$$P_5(x) = \frac{x^2}{2} - \frac{x^3}{4} + \frac{7x^4}{48} - \frac{11x^5}{96}$$

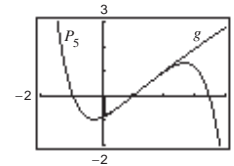


The polynomial is a reasonable approximation on the interval  $(-0.60, 0.73)$ .

65.  $f(x) = \sqrt{x} \ln x, c = 1$

$$P_5(x) = (x-1) - \frac{(x-1)^3}{24} + \frac{(x-1)^4}{24} - \frac{71(x-1)^5}{1920}$$

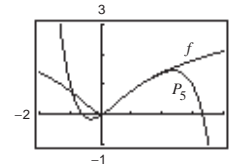
The polynomial is a reasonable approximation on the interval  $[\frac{1}{4}, 2]$ .



66.  $f(x) = \sqrt[3]{x} \cdot \arctan x, c = 1$

$$\begin{aligned} P_5(x) &\approx 0.7854 + 0.7618(x-1) - 0.3412 \left[ \frac{(x-1)^2}{2!} \right] - 0.0424 \left[ \frac{(x-1)^3}{3!} \right] \\ &\quad + 1.3025 \left[ \frac{(x-1)^4}{4!} \right] - 5.5913 \left[ \frac{(x-1)^5}{5!} \right] \end{aligned}$$

The polynomial is a reasonable approximation on the interval  $(0.48, 1.75)$ .



67. See Guidelines, page 680.

68.  $a_{2n+1} = 0$  (odd coefficients are zero)

69. (a) Replace  $x$  with  $(-x)$ .

(c) Multiply series by  $x$ .

(b) Replace  $x$  with  $3x$ .

(d) Replace  $x$  with  $2x$ , then replace  $x$  with  $-2x$ , and add the two together.

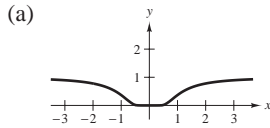
70. The binomial series is  $(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$ . The radius of convergence is  $R = 1$ .

$$\begin{aligned}
 71. \quad y &= \left( \tan \theta - \frac{g}{kv_0 \cos \theta} \right) x - \frac{g}{k^2} \ln \left( 1 - \frac{kx}{v_0 \cos \theta} \right) \\
 &= (\tan \theta)x - \frac{gx}{kv_0 \cos \theta} - \frac{g}{k^2} \left[ -\frac{kx}{v_0 \cos \theta} - \frac{1}{2} \left( \frac{kx}{v_0 \cos \theta} \right)^2 - \frac{1}{3} \left( \frac{kx}{v_0 \cos \theta} \right)^3 - \frac{1}{4} \left( \frac{kx}{v_0 \cos \theta} \right)^4 - \dots \right] \\
 &= (\tan \theta)x - \frac{gx}{kv_0 \cos \theta} + \frac{gx}{kv_0 \cos \theta} + \frac{gx^2}{2v_0^2 \cos^2 \theta} + \frac{gkx^3}{3v_0^3 \cos^3 \theta} + \frac{gk^2x^4}{4v_0^4 \cos^4 \theta} + \dots \\
 &= (\tan \theta)x + \frac{gx^2}{2v_0^2 \cos^2 \theta} + \frac{kgx^3}{3v_0^3 \cos^3 \theta} + \frac{k^2gx^4}{4v_0^4 \cos^4 \theta} + \dots
 \end{aligned}$$

$$72. \quad \theta = 60^\circ, v_0 = 64, k = \frac{1}{16}, g = -32$$

$$\begin{aligned}
 y &= \sqrt{3}x - \frac{32x^2}{2(64)^2(1/2)^2} - \frac{(1/16)(32)x^3}{3(64)^3(1/2)^3} - \frac{(1/16)^2(32)x^4}{4(64)^4(1/2)^4} - \dots \\
 &= \sqrt{3}x - 32 \left[ \frac{2^2x^2}{2(64)^2} + \frac{2^3x^3}{3(64)^316} + \frac{2^4x^4}{4(64)^4(16)^2} + \dots \right] \\
 &= \sqrt{3}x - 32 \sum_{n=2}^{\infty} \frac{2^n x^n}{n(64)^n (16)^{n-2}} = \sqrt{3}x - 32 \sum_{n=2}^{\infty} \frac{x^n}{n(32)^n (16)^{n-2}}
 \end{aligned}$$

$$73. \quad f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$



$$(b) \quad f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{e^{-1/x^2} - 0}{x}$$

Let  $y = \lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x}$ . Then

$$\ln y = \lim_{x \rightarrow 0} \ln \left( \frac{e^{-1/x^2}}{x} \right) = \lim_{x \rightarrow 0^+} \left[ -\frac{1}{x^2} - \ln x \right] = \lim_{x \rightarrow 0^+} \left[ \frac{-1 - x^2 \ln x}{x^2} \right] = -\infty.$$

Thus,  $y = e^{-\infty} = 0$  and we have  $f'(0) = 0$ .

$$(c) \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \dots = 0 \neq f(x) \text{ This series converges to } f \text{ at } x = 0 \text{ only.}$$

$$74. (a) \quad f(x) = \frac{\ln(x^2 + 1)}{x^2}.$$

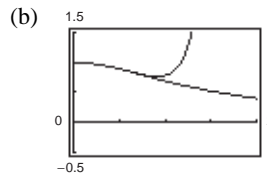
From Exercise 8, you obtain:

$$P = \frac{1}{x^2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n+1}$$

$$P_8 = 1 - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^6}{4} + \frac{x^8}{5}.$$

$$(c) \quad F(x) = \int_0^x \frac{\ln(t^2 + 1)}{t^2} dt$$

$$G(x) = \int_0^x P_8(t) dt$$



|        |        |        |        |        |        |        |
|--------|--------|--------|--------|--------|--------|--------|
| $x$    | 0.25   | 0.50   | 0.75   | 1.00   | 1.50   | 2.00   |
| $F(x)$ | 0.2475 | 0.4810 | 0.6920 | 0.8776 | 1.1798 | 1.4096 |
| $G(x)$ | 0.2475 | 0.4810 | 0.6924 | 0.8865 | 1.6878 | 9.6063 |

(d) The curves are nearly identical for  $0 < x < 1$ . Hence, the integrals nearly agree on that interval.

75. By the Ratio Test:  $\lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{n+1} = 0$  which shows that  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  converges for all  $x$ .

$$\begin{aligned} 76. \ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) \\ &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots\right) \\ &= 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \dots = 2x \sum_{n=0}^{\infty} \frac{x^{2n}}{2n+1}, R=1 \end{aligned}$$

$$\ln 3 = \ln\left(\frac{1+1/2}{1-1/2}\right) \approx 2\left(\frac{1}{2}\right)\left[1 + \frac{(1/2)^2}{3} + \frac{(1/2)^4}{5} + \frac{(1/2)^6}{7}\right] = 1 + \frac{1}{12} + \frac{1}{80} + \frac{1}{448} \approx 1.098065$$

$$(\ln 3 \approx 1.098612)$$

$$77. \binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{3!} = \frac{60}{6} = 10$$

$$78. \binom{-2}{2} = \frac{(-2)(-3)}{2!} = 3$$

$$\begin{aligned} 79. \binom{0.5}{4} &= \frac{(0.5)(-0.5)(-1.5)(-2.5)}{4!} \\ &= -0.0390625 = -\frac{5}{128} \end{aligned}$$

$$\begin{aligned} 80. \binom{-1/3}{5} &= \frac{(-1/3)(-4/3)(-7/3)(-10/3)(-13/3)}{5!} \\ &= \frac{-91}{729} \approx -0.12483 \end{aligned}$$

$$81. (1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

$$\text{Example: } (1+x)^2 = \sum_{n=0}^{\infty} \binom{2}{n} x^n = 1 + 2x + x^2$$

82. Assume  $e = p/q$  is rational. Let  $N > q$  and form the following.

$$e - \left[1 + 1 + \frac{1}{2!} + \dots + \frac{1}{N!}\right] = \frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \dots$$

Set  $a = N! \left[ e - \left(1 + 1 + \dots + \frac{1}{N!}\right) \right]$ , a positive integer. But,

$$\begin{aligned} a &= N! \left[ \frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \dots \right] = \frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \dots < \frac{1}{N+1} + \frac{1}{(N+1)^2} + \dots \\ &= \frac{1}{N+1} \left[ 1 + \frac{1}{N+1} + \frac{1}{(N+1)^2} + \dots \right] = \frac{1}{N+1} \left[ \frac{1}{1 - \left(\frac{1}{N+1}\right)} \right] = \frac{1}{N}, \text{ a contradiction.} \end{aligned}$$

$$83. g(x) = \frac{x}{1-x-x^2} = a_0 + a_1x + a_2x^2 + \dots$$

$$x = (1-x-x^2)(a_0 + a_1x + a_2x^2 + \dots)$$

$$x = a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 + (a_3 - a_2 - a_1)x^3 + \dots$$

Equating coefficients,

$$a_0 = 0$$

$$a_1 - a_0 = 1 \Rightarrow a_1 = 1$$

$$a_2 - a_1 - a_0 = 0 \Rightarrow a_2 = 1$$

$$a_3 - a_2 - a_1 = 0 \Rightarrow a_3 = 2$$

$$a_4 = a_3 + a_2 = 3, \text{ etc.}$$

In general,  $a_n = a_{n-1} + a_{n-2}$ . The coefficients are the Fibonacci numbers.

84. Assume the interval is  $[-1, 1]$ . Let  $x \in [-1, 1]$ ,

$$f(1) = f(x) + (1-x)f'(x) + \frac{1}{2}(1-x)^2 f''(c), \quad c \in (x, 1)$$

$$f(-1) = f(x) + (-1-x)f'(x) + \frac{1}{2}(-1-x)^2 f''(d), \quad d \in (-1, x)$$

$$\text{Hence, } f(1) - f(-1) = 2f'(x) + \frac{1}{2}(1-x)^2 f''(c) - \frac{1}{2}(1+x)^2 f''(d)$$

$$2f'(x) = f(1) - f(-1) - \frac{1}{2}(1-x)^2 f''(c) + \frac{1}{2}(1+x)^2 f''(d)$$

Since  $|f(x)| \leq 1$  and  $|f''(x)| \leq 1$ ,

$$\begin{aligned} 2|f'(x)| &\leq |f(1)| + |f(-1)| + \frac{1}{2}(1-x)^2 |f''(c)| + \frac{1}{2}(1+x)^2 |f''(d)| \\ &\leq 1 + 1 + \frac{1}{2}(1-x^2) + \frac{1}{2}(1+x)^2 \\ &= 3 + x^2 \leq 4. \end{aligned}$$

Thus,  $|f'(x)| \leq 2$ .

**Note:** Let  $f(x) = \frac{1}{2}(x+1)^2 - 1$ . Then  $|f'(x)| \leq 1$ ,  $|f''(x)| = 1$  and  $f'(1) = 2$ .

## Review Exercises for Chapter 9

1.  $a_n = \frac{1}{n!}$

2.  $a_n = \frac{n}{n^2 + 1}$

3.  $a_n = 4 + \frac{2}{n}$ : 6, 5, 4.67, . . .

Matches (a)

4.  $a_n = 4 - \frac{n}{2}$ : 3.5, 3, . . .

Matches (c)

5.  $a_n = 10(0.3)^{n-1}$ : 10, 3, . . .

Matches (d)

6.  $a_n = 6\left(-\frac{2}{3}\right)^{n-1}$ : 6, -4, . . .

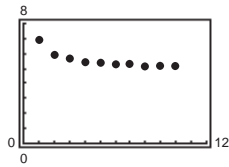
Matches (b)

7.  $a_n = \frac{5n+2}{n}$

The sequence seems to converge to 5.

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{5n+2}{n}$$

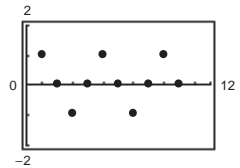
$$= \lim_{n \rightarrow \infty} \left(5 + \frac{2}{n}\right) = 5$$



8.  $a_n = \sin \frac{n\pi}{2}$

The sequence seems to diverge (oscillates).

$$\sin \frac{n\pi}{2}: 1, 0, -1, 0, 1, 0, \dots$$



9.  $\lim_{n \rightarrow \infty} \frac{n+1}{n^2} = 0$

Converges

10.  $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$

Converges

11.  $\lim_{n \rightarrow \infty} \frac{n^3}{n^2+1} = \infty$

12.  $\lim_{n \rightarrow \infty} \frac{n}{\ln(n)} = \lim_{n \rightarrow \infty} \frac{1}{1/n}$

$$= \infty$$

Diverges

13.  $\lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) = \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$  Converges

14.  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2n}\right)^n = \lim_{k \rightarrow \infty} \left[\left(1 + \frac{1}{k}\right)^k\right]^{1/2} = e^{1/2}$

Converges;  $k = 2n$

15.  $\lim_{n \rightarrow \infty} \frac{\sin \sqrt{n}}{\sqrt{n}} = 0$

Converges



16. Let  $y = (b^n + c^n)^{1/n}$

$$\ln y = \frac{\ln(b^n + c^n)}{n}$$

$$\lim_{n \rightarrow \infty} \ln y = \lim_{n \rightarrow \infty} \frac{1}{b^n + c^n} (b^n \ln b + c^n \ln c).$$

Assume  $b \geq c$  and note that the terms

$$\frac{b^n \ln b + c^n \ln c}{b^n + c^n} = \frac{b^n \ln b}{b^n + c^n} + \frac{c^n \ln c}{b^n + c^n}$$

converge as  $n \rightarrow \infty$ . Hence  $a_n$  converges.

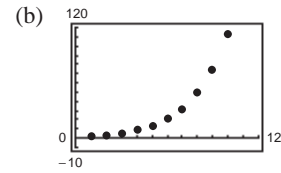
18. (a)  $V_n = 120,000(0.70)^n, n = 1, 2, 3, 4, 5$

(b)  $V_5 = 120,000(0.70)^5 = \$20,168.40$

19. (a)

|       |      |       |       |        |          |
|-------|------|-------|-------|--------|----------|
| $k$   | 5    | 10    | 15    | 20     | 25       |
| $S_k$ | 13.2 | 113.3 | 873.8 | 6448.5 | 50,500.3 |

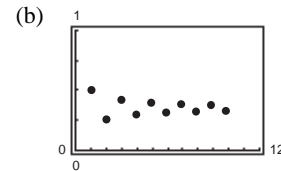
The series diverges (geometric  $r = \frac{3}{2} > 1$ ).



20. (a)

|       |        |        |        |        |        |
|-------|--------|--------|--------|--------|--------|
| $k$   | 5      | 10     | 15     | 20     | 25     |
| $S_k$ | 0.3917 | 0.3228 | 0.3627 | 0.3344 | 0.3564 |

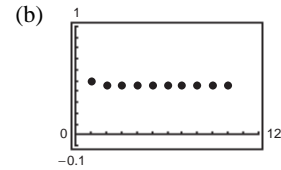
The series converges by the Alternating Series Test.



21. (a)

|       |        |        |        |        |        |
|-------|--------|--------|--------|--------|--------|
| $k$   | 5      | 10     | 15     | 20     | 25     |
| $S_k$ | 0.4597 | 0.4597 | 0.4597 | 0.4597 | 0.4597 |

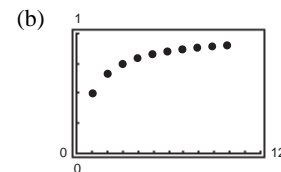
The series converges by the Alternating Series Test.



22. (a)

|       |        |        |        |        |        |
|-------|--------|--------|--------|--------|--------|
| $k$   | 5      | 10     | 15     | 20     | 25     |
| $S_k$ | 0.8333 | 0.9091 | 0.9375 | 0.9524 | 0.9615 |

The series converges, by the Limit Comparison Test with  $\sum \frac{1}{n^2}$ .



23. Converges. Geometric series,  $r = 0.82, |r| < 1$ .

24. Diverges. Geometric series,  $r = 1.82 > 1$ .

25. Diverges.  $n$ th-Term Test.  $\lim_{n \rightarrow \infty} a_n \neq 0$ .

26. Diverges.  $n$ th-Term Test,  $\lim_{n \rightarrow \infty} a_n = \frac{2}{3}$ .

27.  $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$  Geometric series with  $a = 1$  and  $r = \frac{2}{3}$ .

$$S = \frac{a}{1-r} = \frac{1}{1-(2/3)} = \frac{1}{1/3} = 3$$

28.  $\sum_{n=0}^{\infty} \frac{2^{n+2}}{3^n} = 4 \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 4(3) = 12$

See Exercise 27.

17.  $A_n = 5000 \left(1 + \frac{0.05}{4}\right)^n = 5000(1.0125)^n$

$n = 1, 2, 3$

(a)  $A_1 \approx 5062.50$        $A_5 \approx 5320.41$

$A_2 \approx 5125.78$        $A_6 \approx 5386.92$

$A_3 \approx 5189.85$        $A_7 \approx 5454.25$

$A_4 \approx 5254.73$        $A_8 \approx 5522.43$

(b)  $A_{40} \approx 8218.10$

$$29. \sum_{n=0}^{\infty} \left( \frac{1}{2^n} - \frac{1}{3^n} \right) = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n - \sum_{n=0}^{\infty} \left( \frac{1}{3} \right)^n = \frac{1}{1 - (1/2)} - \frac{1}{1 - (1/3)} = 2 - \frac{3}{2} = \frac{1}{2}$$

$$30. \sum_{n=0}^{\infty} \left[ \left( \frac{2}{3} \right)^n - \frac{1}{(n+1)(n+2)} \right] = \sum_{n=0}^{\infty} \left( \frac{2}{3} \right)^n - \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+2} \right) \\ = \frac{1}{1 - (2/3)} - \left[ \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \cdots \right] = 3 - 1 = 2$$

$$31. (a) 0.\overline{09} = 0.09 + 0.0009 + 0.000009 + \cdots = 0.09(1 + 0.01 + 0.0001 + \cdots) = \sum_{n=0}^{\infty} (0.09)(0.01)^n$$

$$(b) 0.\overline{09} = \frac{0.09}{1 - 0.01} = \frac{1}{11}$$

$$32. (a) \overline{0.923076} = 0.923076[1 + 0.000001 + (0.000001)^2 + \cdots] = \sum_{n=0}^{\infty} (0.923076)(0.000001)^n$$

$$(b) \overline{0.923076} = \frac{0.923076}{1 - 0.000001} = \frac{923,076}{999,999} = \frac{12(76,923)}{13(76,923)} = \frac{12}{13}$$

$$33. D_1 = 8$$

$$D_2 = 0.7(8) + 0.7(8) = 16(0.7)$$

$$\vdots$$

$$D = 8 + 16(0.7) + 16(0.7)^2 + \cdots + 16(0.7)^n + \cdots$$

$$= -8 + \sum_{n=0}^{\infty} 16(0.7)^n = -8 + \frac{16}{1 - 0.7} = 45\frac{1}{3} \text{ meters}$$

$$34. S = \sum_{n=0}^{39} 32,000(1.055)^n = \frac{32,000(1 - 1.055^{40})}{1 - 1.055}$$

$$\approx \$4,371,379.65$$

35. See Exercise 110 in Section 9.2.

$$A = \frac{P(e^{rt} - 1)}{e^{r/12} - 1} \\ = \frac{200(e^{(0.06)(2)} - 1)}{e^{0.06/12} - 1} \\ \approx \$5087.14$$

36. See Exercise 110 in Section 9.2.

$$A = P \left( \frac{12}{r} \right) \left[ \left( 1 + \frac{r}{12} \right)^{12t} - 1 \right] \\ = 100 \left( \frac{12}{0.035} \right) \left[ \left( 1 + \frac{0.035}{12} \right)^{120} - 1 \right] \\ \approx \$14,343.25$$

$$37. \int_1^{\infty} x^{-4} \ln(x) dx = \lim_{b \rightarrow \infty} \left[ -\frac{\ln x}{3x^3} - \frac{1}{9x^3} \right]_1^b = 0 + \frac{1}{9} = \frac{1}{9}$$

By the Integral Test, the series converges.

$$38. \sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$$

Divergent  $p$ -series,  $p = \frac{3}{4} < 1$

$$39. \sum_{n=1}^{\infty} \left( \frac{1}{n^2} - \frac{1}{n} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n}$$

Since the second series is a divergent  $p$ -series while the first series is a convergent  $p$ -series, the difference diverges.

$$40. \sum_{n=1}^{\infty} \left( \frac{1}{n^2} - \frac{1}{2^n} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{2^n}$$

The first series is a convergent  $p$ -series and the second series is a convergent geometric series. Therefore, their difference converges.

$$41. \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 2n}}$$

$$\lim_{n \rightarrow \infty} \frac{1/\sqrt{n^3 + 2n}}{1/(n^{3/2})} = \lim_{n \rightarrow \infty} \frac{n^{3/2}}{\sqrt{n^3 + 2n}} = 1$$

By a limit comparison test with the convergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}, \text{ the series converges.}$$

$$42. \sum_{n=1}^{\infty} \frac{n+1}{n(n+2)}$$

$$\lim_{n \rightarrow \infty} \frac{(n+1)/n(n+2)}{1/n} = \lim_{n \rightarrow \infty} \frac{n+1}{n+2} = 1$$

By a limit comparison test with  $\sum_{n=1}^{\infty} \frac{1}{n}$ , the series diverges.

$$43. \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

$$= \left( \frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2n-1}{2n-2} \right) \frac{1}{2n} > \frac{1}{2n}$$

Since  $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$  diverges (harmonic series),  
so does the original series.

45. Converges by the Alternating Series Test.  
(Conditional convergence)

47. Diverges by the  $n$ th-Term Test.

$$49. \sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n+1}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{e^{n^2}(n+1)}{e^{n^2+2n+1}n} \right|$$

$$= \lim_{n \rightarrow \infty} \left( \frac{1}{e^{2n+1}} \right) \left( \frac{n+1}{n} \right)$$

$$= (0)(1) = 0 < 1$$

By the Ratio Test, the series converges.

$$51. \sum_{n=1}^{\infty} \frac{2^n}{n^3}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^n} \right| = \lim_{n \rightarrow \infty} \frac{2n^3}{(n+1)^3} = 2$$

Therefore, by the Ratio Test, the series diverges.

$$52. \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1 \cdot 3 \cdots (2n-1)(2n+1)}{2 \cdot 5 \cdots (3n-1)(3n+2)} \cdot \frac{2 \cdot 5 \cdots (3n-1)}{1 \cdot 3 \cdots (2n-1)} \right| = \lim_{n \rightarrow \infty} \frac{2n+1}{3n+2} = \frac{2}{3}$$

By the Ratio Test, the series converges.

44. Since  $\sum_{n=1}^{\infty} \frac{1}{3^n}$  converges,  $\sum_{n=1}^{\infty} \frac{1}{3^n - 5}$  converges by the  
Limit Comparison Test.

$$46. \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$$

$$a_{n+1} = \frac{\sqrt{n+1}}{n+2} \leq \frac{\sqrt{n}}{n+1} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n+1} = 0$$

By the Alternating Series Test, the series converges.

48. Converges by the Alternating Series Test.

$$a_{n+1} = \frac{3 \ln(n+1)}{n+1} < \frac{3 \ln n}{n} = a_n, \lim_{n \rightarrow \infty} \frac{3 \ln n}{n} = 0$$

$$50. \sum_{n=1}^{\infty} \frac{n!}{e^n}$$

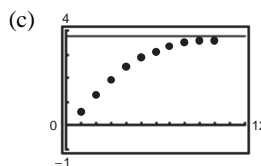
$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{n+1}{e} = \infty$$

By the Ratio Test, the series diverges.

53. (a) Ratio Test:  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)(3/5)^{n+1}}{n(3/5)^n} = \lim_{n \rightarrow \infty} \left( \frac{n+1}{n} \right) \left( \frac{3}{5} \right) = \frac{3}{5} < 1$ , Converges

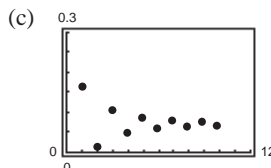
|     |       |        |        |        |        |        |
|-----|-------|--------|--------|--------|--------|--------|
| (b) | $x$   | 5      | 10     | 15     | 20     | 25     |
|     | $S_n$ | 2.8752 | 3.6366 | 3.7377 | 3.7488 | 3.7499 |



(d) The sum is approximately 3.75.

54. (a) The series converges by the Alternating Series Test.

|     |       |        |        |        |        |        |
|-----|-------|--------|--------|--------|--------|--------|
| (b) | $x$   | 5      | 10     | 15     | 20     | 25     |
|     | $S_n$ | 0.0871 | 0.0669 | 0.0734 | 0.0702 | 0.0721 |



(d) The sum is approximately 0.0714.

55. (a)  $\int_N^\infty \frac{1}{x^2} dx = \left[ -\frac{1}{x} \right]_N^\infty = \frac{1}{N}$

|                                  |        |        |        |        |        |
|----------------------------------|--------|--------|--------|--------|--------|
| $N$                              | 5      | 10     | 20     | 30     | 40     |
| $\sum_{n=1}^N \frac{1}{n^2}$     | 1.4636 | 1.5498 | 1.5962 | 1.6122 | 1.6202 |
| $\int_N^\infty \frac{1}{x^2} dx$ | 0.2000 | 0.1000 | 0.0500 | 0.0333 | 0.0250 |

(b)  $\int_N^\infty \frac{1}{x^5} dx = \left[ -\frac{1}{4x^4} \right]_N^\infty = \frac{1}{4N^4}$

|                                  |        |        |        |        |        |
|----------------------------------|--------|--------|--------|--------|--------|
| $N$                              | 5      | 10     | 20     | 30     | 40     |
| $\sum_{n=1}^N \frac{1}{n^5}$     | 1.0367 | 1.0369 | 1.0369 | 1.0369 | 1.0369 |
| $\int_N^\infty \frac{1}{x^5} dx$ | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

The series in part (b) converges more rapidly. The integral values represent the remainders of the partial sums.

56. No. Let  $a_n = \frac{3937.5}{n^2}$ , then  $a_{75} = 0.7$ . The series  $\sum_{n=1}^{\infty} \frac{3937.5}{n^2}$  is a convergent  $p$ -series.

57.  $f(x) = e^{-x/2}$        $f(0) = 1$        $P_3(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$

$f'(x) = -\frac{1}{2}e^{-x/2}$        $f'(0) = -\frac{1}{2}$        $= 1 - \frac{1}{2}x + \frac{1}{4} \frac{x^2}{2!} - \frac{1}{8} \frac{x^3}{3!}$

$f''(x) = \frac{1}{4}e^{-x/2}$        $f''(0) = \frac{1}{4}$        $= 1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3$

$f'''(x) = -\frac{1}{8}e^{-x/2}$        $f'''(0) = -\frac{1}{8}$

58.  $f(x) = \tan x$        $f\left(-\frac{\pi}{4}\right) = -1$        $P_3(x) = -1 + 2\left(x + \frac{\pi}{4}\right) - 2\left(x + \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x + \frac{\pi}{4}\right)^3$

$f'(x) = \sec^2 x$        $f'\left(-\frac{\pi}{4}\right) = 2$

$f''(x) = 2 \sec^2 x \tan x$        $f''\left(-\frac{\pi}{4}\right) = -4$

$f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x$        $f'''\left(-\frac{\pi}{4}\right) = 16$

59. Since  $\frac{(95\pi)^9}{180^9 \cdot 9!} < 0.001$ , use four terms.

$$\sin 95^\circ = \sin\left(\frac{95\pi}{180}\right) \approx \frac{95\pi}{180} - \frac{(95\pi)^3}{180^3 3!} + \frac{(95\pi)^5}{180^5 5!} - \frac{(95\pi)^7}{180^7 7!} \approx 0.99594$$

60.  $\cos(0.75) \approx 1 - \frac{(0.75)^2}{2!} + \frac{(0.75)^4}{4!} - \frac{(0.75)^6}{6!} \approx 0.7317$

61.  $\ln(1.75) \approx (0.75) - \frac{(0.75)^2}{2} + \frac{(0.75)^3}{3} - \frac{(0.75)^4}{4} + \frac{(0.75)^5}{5} - \frac{(0.75)^6}{6} + \dots - \frac{(0.75)^{14}}{14} \approx 0.559062$

62.  $e^{-0.25} \approx 1 - 0.25 + \frac{(0.25)^2}{2!} - \frac{(0.25)^3}{3!} + \frac{(0.25)^4}{4!} \approx 0.779$

63.  $f(x) = \cos x$ ,  $c = 0$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$$

$$|f^{(n+1)}(z)| \leq 1 \Rightarrow R_n(x) \leq \frac{x^{n+1}}{(n+1)!}$$

(a)  $R_n(x) \leq \frac{(0.5)^{n+1}}{(n+1)!} < 0.001$

This inequality is true for  $n = 4$ .

(b)  $R_n(x) \leq \frac{(1)^{n+1}}{(n+1)!} < 0.001$

This inequality is true for  $n = 6$ .

(c)  $R_n(x) \leq \frac{(0.5)^{n+1}}{(n+1)!} < 0.0001$

This inequality is true for  $n = 5$ .

(d)  $R_n(x) \leq \frac{2^{n+1}}{(n+1)!} < 0.0001$

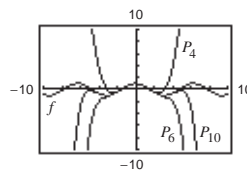
This inequality is true for  $n = 10$ .

64.  $f(x) = \cos x$

$$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$P_{10}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$$



65.  $\sum_{n=0}^{\infty} \left(\frac{x}{10}\right)^n$

Geometric series which converges only if  $|x/10| < 1$  or  $-10 < x < 10$ .

67.  $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{(n+1)^2}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (x-2)^{n+1}}{(n+2)^2} \cdot \frac{(n+1)^2}{(-1)^n (x-2)^n} \right| = |x-2|$$

$$R = 1$$

Center: 2

Since the series converges when  $x = 1$  and when  $x = 3$ , the interval of convergence is  $1 \leq x \leq 3$ .

66.  $\sum_{n=0}^{\infty} (2x)^n$

Geometric series which converges only if  $|2x| < 1$  or  $-\frac{1}{2} < x < \frac{1}{2}$ .

68.  $\sum_{n=1}^{\infty} \frac{3^n (x-2)^n}{n}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1} (x-2)^{n+1}}{n+1} \cdot \frac{n}{3^n (x-2)^n} \right| = 3|x-2|$$

$$R = \frac{1}{3}$$

Center: 2

Since the series converges at  $\frac{5}{3}$  and diverges at  $\frac{7}{3}$ , the interval of convergence is  $\frac{5}{3} \leq x < \frac{7}{3}$ .

$$69. \sum_{n=0}^{\infty} n!(x-2)^n$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!(x-2)^{n+1}}{n!(x-2)^n} \right| = \infty$$

which implies that the series converges only at the center  $x = 2$ .

$$70. \sum_{n=0}^{\infty} \frac{(x-2)^n}{2^n} = \sum_{n=0}^{\infty} \left( \frac{x-2}{2} \right)^n$$

Geometric series which converges only if

$$\left| \frac{x-2}{2} \right| < 1 \quad \text{or} \quad 0 < x < 4.$$

$$71. \quad y = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^n (n!)^2}$$

$$y' = \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{4^n (n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2) x^{2n+1}}{4^{n+1} [(n+1)!]^2}$$

$$y'' = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2)(2n+1) x^{2n}}{4^{n+1} [(n+1)!]^2}$$

$$\begin{aligned} x^2 y'' + xy' + x^2 y &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2)(2n+1) x^{2n+2}}{4^{n+1} [(n+1)!]^2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2) x^{2n+2}}{4^{n+1} [(n+1)!]^2} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{4^n (n!)^2} \\ &= \sum_{n=0}^{\infty} \left[ (-1)^{n+1} \frac{(2n+2)(2n+1)}{4^{n+1} [(n+1)!]^2} + \frac{(-1)^{n+1} (2n+2)}{4^{n+1} [(n+1)!]^2} + \frac{(-1)^n}{4^n (n!)^2} \right] x^{2n+2} \\ &= \sum_{n=0}^{\infty} \left[ \frac{(-1)^{n+1} (2n+2)(2n+1+1)}{4^{n+1} [(n+1)!]^2} + (-1)^n \frac{1}{4^n (n!)^2} \right] x^{2n+2} \\ &= \sum_{n=0}^{\infty} \left[ \frac{(-1)^{n+1} 4(n+1)^2}{4^{n+1} [(n+1)!]^2} + (-1)^n \frac{1}{4^n (n!)^2} \right] x^{2n+2} \\ &= \sum_{n=0}^{\infty} \left[ \frac{(-1)^{n+1} 1}{4^n (n!)^2} + (-1)^n \frac{1}{4^n (n!)^2} \right] x^{2n+2} = 0 \end{aligned}$$

$$72. \quad y = \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{2^n n!}$$

$$y' = \sum_{n=1}^{\infty} \frac{(-3)^n (2n) x^{2n-1}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2) x^{2n+1}}{2^{n+1} (n+1)!}$$

$$y'' = \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2)(2n+1) x^{2n}}{2^{n+1} (n+1)!}$$

$$\begin{aligned} y'' + 3xy' + 3y &= \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2)(2n+1) x^{2n}}{2^{n+1} (n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} (2n+2) x^{2n+2}}{2^{n+1} (n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+1} (2n+2) x^{2n}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} [-(2n+1) + 1] + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} (-2n) + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^{n+1} x^{2n}}{2^n n!} (2n) + \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^{n-1} (n-1)!} \cdot \frac{2n}{2n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} [-2n + 2n] = 0 \end{aligned}$$

$$73. \quad \frac{2}{3-x} = \frac{2/3}{1-(x/3)} = \frac{a}{1-r}$$

$$\sum_{n=0}^{\infty} \frac{2}{3} \left( \frac{x}{3} \right)^n = \sum_{n=0}^{\infty} \frac{2x^n}{3^{n+1}}$$

$$74. \quad \frac{3}{2+x} = \frac{3/2}{1+(x/2)} = \frac{3/2}{1-(-x/2)} = \frac{a}{1-r}$$

$$\sum_{n=0}^{\infty} \frac{3}{2} \left( -\frac{x}{2} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3x^n}{2^{n+1}}$$

75.  $g(x) = \frac{2}{3-x}$ . Power series  $\sum_{n=0}^{\infty} \frac{2}{3} \left(\frac{x}{3}\right)^n$

Derivative:  $\sum_{n=1}^{\infty} \frac{2}{3} n \left(\frac{x}{3}\right)^{n-1} \left(\frac{1}{3}\right) = \sum_{n=1}^{\infty} \frac{2}{9} n \left(\frac{x}{3}\right)^{n-1}$   
 $= \sum_{n=0}^{\infty} \frac{2}{9} (n+1) \left(\frac{x}{3}\right)^n$

76. Integral:  $\sum_{n=0}^{\infty} \frac{(-1)^n 3x^{n+1}}{(n+1)2^{n+1}}$

77.  $1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \cdots = \sum_{n=0}^{\infty} \left(\frac{2x}{3}\right)^n = \frac{1}{1-(2x/3)} = \frac{3}{3-2x}$ ,  $-\frac{3}{2} < x < \frac{3}{2}$

78.  $8 - 2(x-3) + \frac{1}{2}(x-3)^2 - \frac{1}{8}(x-3)^3 + \cdots = \sum_{n=0}^{\infty} 8 \left[\frac{-(x-3)}{4}\right]^n = \frac{8}{1-[-(x-3)/4]}$   
 $= \frac{32}{4+(x-3)} = \frac{32}{1+x}$ ,  $-1 < x < 7$

79.  $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x, \cdots$

$\sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)[x - (3\pi/4)]^n}{n!}$   
 $= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \left(x - \frac{3\pi}{4}\right) - \frac{\sqrt{2}}{2 \cdot 2!} \left(x - \frac{3\pi}{4}\right)^2 + \cdots = \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2} [x - (3\pi/4)]^n}{n!}$

80.  $f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(-\pi/4)[x + (\pi/4)]^n}{n!} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \left(x + \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2 \cdot 2!} \left(x + \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2 \cdot 3!} \left(x + \frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{2 \cdot 4!} \left(x + \frac{\pi}{4}\right)^4 + \cdots$   
 $= \frac{\sqrt{2}}{2} \left[ 1 + \left(x + \frac{\pi}{4}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{[n(n+1)]/2} [x + (\pi/4)]^{n+1}}{(n+1)!} \right]$

81.  $3^x = (e^{\ln 3})^x = e^{x \ln 3}$  and since  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ , we have  $3^x = \sum_{n=0}^{\infty} \frac{(x \ln 3)^n}{n!} = 1 + x \ln 3 + \frac{x^2 [\ln 3]^2}{2!} + \frac{x^3 [\ln 3]^3}{3!} + \frac{x^4 [\ln 3]^4}{4!} + \cdots$

82.  $f(x) = \csc(x)$

$f'(x) = -\csc(x) \cot(x)$

$f''(x) = \csc^3(x) + \csc(x) \cot^2(x)$

$f'''(x) = -5 \csc^3(x) \cot(x) - \csc(x) \cot^3(x)$

$f^{(4)}(x) = 5 \csc^5(x) + 15 \csc^3(x) \cot^2(x) + \csc(x) \cot^4(x)$

$\csc(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)[x - (\pi/2)]^n}{n!} = 1 + \frac{1}{2!} \left(x - \frac{\pi}{2}\right)^2 + \frac{5}{4!} \left(x - \frac{\pi}{2}\right)^4 + \cdots$

$$83. f(x) = \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2}$$

$$f''(x) = \frac{2}{x^3}$$

$$f'''(x) = -\frac{6}{x^4}, \dots$$

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(-1)(x+1)^n}{n!} = \sum_{n=0}^{\infty} \frac{-n!(x+1)^n}{n!} = -\sum_{n=0}^{\infty} (x+1)^n, -2 < x < 0$$

$$84. f(x) = x^{1/2}$$

$$f'(x) = \frac{1}{2}x^{-1/2}$$

$$f''(x) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x^{-3/2}$$

$$f'''(x) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)x^{-5/2}$$

$$f^{(4)}(x) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)x^{-7/2}, \dots$$

$$\begin{aligned} \sqrt{x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(4)(x-4)^n}{n!} = 2 + \frac{(x-4)}{2^2} - \frac{(x-4)^2}{2^5 2!} + \frac{1 \cdot 3(x-4)^3}{2^8 3!} - \frac{1 \cdot 3 \cdot 5(x-4)^4}{2^{11} 4!} + \dots \\ &= 2 + \frac{(x-4)}{2^2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)(x-4)^n}{2^{3n-1} n!} \end{aligned}$$

$$85. (1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \dots$$

$$\begin{aligned} (1+x)^{1/5} &= 1 + \frac{x}{5} + \frac{(1/5)(-4/5)x^2}{2!} + \frac{1/5(-4/5)(-9/5)x^3}{3!} + \dots \\ &= 1 + \frac{1}{5}x - \frac{1 \cdot 4x^2}{5^2 2!} + \frac{1 \cdot 4 \cdot 9x^3}{5^3 3!} - \dots \\ &= 1 + \frac{x}{5} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 4 \cdot 9 \cdot 14 \cdots (5n-6)x^n}{5^n n!} \\ &= 1 + \frac{x}{5} - \frac{2}{25}x^2 + \frac{6}{125}x^3 - \dots \end{aligned}$$

$$86. h(x) = (1+x)^{-3}$$

$$h'(x) = -3(1+x)^{-4}$$

$$h''(x) = 12(1+x)^{-5}$$

$$h'''(x) = -60(1+x)^{-6}$$

$$h^{(4)}(x) = 360(1+x)^{-7}$$

$$h^{(5)}(x) = -2520(1+x)^{-8}$$

$$\frac{1}{(1+x)^3} = 1 - 3x + \frac{12x^2}{2!} - \frac{60x^3}{3!} + \frac{360x^4}{4!} - \frac{2520x^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)! x^n}{2n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1)x^n}{2}$$



$$87. \ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad 0 < x \leq 2$$

$$\begin{aligned} \ln\left(\frac{5}{4}\right) &= \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{(5/4)-1}{n}\right)^n \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{4^n n} \approx 0.2231 \end{aligned}$$

$$88. \ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad 0 < x \leq 2$$

$$\begin{aligned} \ln\left(\frac{6}{5}\right) &= \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{(6/5)-1}{n}\right)^n \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5^n n} \approx 0.1823 \end{aligned}$$

$$89. e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$$

$$e^{1/2} = \sum_{n=0}^{\infty} \frac{(1/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{2^n n!} \approx 1.6487$$

$$90. e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$$

$$e^{2/3} = \sum_{n=0}^{\infty} \frac{(2/3)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{3^n n!} \approx 1.9477$$

$$91. \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty$$

$$\cos\left(\frac{2}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{3^{2n}(2n)!} \approx 0.7859$$

$$92. \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$$

$$\sin\left(\frac{1}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)!} \approx 0.3272$$

93. The series for Exercise 41 converges very slowly because the terms approach 0 at a slow rate.

$$94. \frac{1}{\sqrt{1+x^3}} = (1+x^3)^{-1/2}, \quad k = -\frac{1}{2}$$

$$= 1 - \frac{1}{2}(x^3) + \frac{(-1/2)(-3/2)}{2!}x^6 + \frac{(-1/2)(-3/2)(-5/2)}{3!}x^9 + \dots$$

$$= 1 - \frac{x^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot \dots \cdot (2n-1)}{2^n n!} x^{3n}$$

$$95. (a) f(x) = e^{2x} \quad f(0) = 1$$

$$f'(x) = 2e^{2x} \quad f'(0) = 2$$

$$f''(x) = 4e^{2x} \quad f''(0) = 4$$

$$f'''(x) = 8e^{2x} \quad f'''(0) = 8$$

$$P(x) = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

$$(b) e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

$$P(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

$$(c) e^x \cdot e^x = \left(1 + x + \frac{x^2}{2!} + \dots\right) \left(1 + x + \frac{x^2}{2!} + \dots\right)$$

$$P = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

$$96. (a) f(x) = \sin 2x \quad f(0) = 0$$

$$f'(x) = 2 \cos 2x \quad f'(0) = 2$$

$$f''(x) = -4 \sin 2x \quad f''(0) = 0$$

$$f'''(x) = -8 \cos 2x \quad f'''(0) = -8$$

$$f^{(4)}(x) = 16 \sin 2x \quad f^{(4)}(0) = 0$$

$$f^{(5)}(x) = 32 \cos 2x \quad f^{(5)}(0) = 32$$

$$f^{(6)}(x) = -64 \sin 2x \quad f^{(6)}(0) = 0$$

$$f^{(7)}(x) = -128 \cos 2x \quad f^{(7)}(0) = -128$$

$$\sin 2x = 0 + 2x + \frac{0x^2}{2!} - \frac{8x^3}{3!} + \frac{0x^4}{4!} + \frac{32x^5}{5!} + \frac{0x^6}{6!} - \frac{128x^7}{7!} + \dots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \dots$$

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## 96. —CONTINUED—

$$\begin{aligned} \text{(b) } \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ \sin 2x &= \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots \\ &= 2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \frac{128x^7}{5040} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots \end{aligned}$$

$$\begin{aligned} \text{(c) } \sin 2x &= 2 \sin x \cos x \\ &= 2 \left( x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \right) \left( 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \right) \\ &= 2 \left[ x + \left( -\frac{x^3}{2} - \frac{x^3}{6} \right) + \left( \frac{x^5}{24} + \frac{x^5}{12} + \frac{x^5}{120} \right) + \left( -\frac{x^7}{720} - \frac{x^7}{144} - \frac{x^7}{240} - \frac{x^7}{5040} \right) + \cdots \right] \\ &= 2 \left[ x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315} + \cdots \right] = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots \end{aligned}$$

$$97. \quad \sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

$$\frac{\sin t}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}$$

$$\begin{aligned} \int_0^x \frac{\sin t}{t} dt &= \left[ \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)(2n+1)!} \right]_0^x \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} \end{aligned}$$

$$98. \quad \cos t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}$$

$$\cos \frac{\sqrt{t}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{2^{2n}(2n)!}$$

$$\begin{aligned} \int_0^x \cos \frac{\sqrt{t}}{2} dt &= \left[ \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{2^{2n}(2n)!(n+1)!} \right]_0^x \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{2^{2n}(2n)!(n+1)!} \end{aligned}$$

$$99. \quad \frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n$$

$$\ln(1+t) = \int \frac{1}{1+t} dt = \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{n+1}$$

$$\frac{\ln(t+1)}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n+1}$$

$$\int_0^x \frac{\ln(t+1)}{t} dt = \left[ \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{(n+1)^2} \right]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)^2}$$

$$100. \quad e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

$$e^t - 1 = \sum_{n=1}^{\infty} \frac{t^n}{n!}$$

$$\frac{e^t - 1}{t} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!}$$

$$\int_0^x \frac{e^t - 1}{t} dt = \left[ \sum_{n=1}^{\infty} \frac{t^n}{n \cdot n!} \right]_0^x = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$$

$$101. \quad \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

$$\frac{\arctan x}{\sqrt{x}} = \sqrt{x} - \frac{x^{5/2}}{3} + \frac{x^{9/2}}{5} - \frac{x^{13/2}}{7} + \frac{x^{17/2}}{9} - \cdots$$

$$\lim_{x \rightarrow 0^+} \frac{\arctan x}{\sqrt{x}} = 0$$

$$\text{By L'Hôpital's Rule, } \lim_{x \rightarrow 0^+} \frac{\arctan x}{\sqrt{x}} = \lim_{x \rightarrow 0^+} \frac{\left( \frac{1}{1+x^2} \right)}{\left( \frac{1}{2\sqrt{x}} \right)} = \lim_{x \rightarrow 0^+} \frac{2\sqrt{x}}{1+x^2} = 0.$$

$$102. \quad \arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots$$

$$\frac{\arcsin x}{x} = 1 + \frac{x^2}{2 \cdot 3} + \frac{1 \cdot 3x^4}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^6}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots$$

$$\lim_{x \rightarrow 0} \frac{\arcsin x}{x} = 1$$

$$\text{By L'Hôpital's Rule, } \lim_{x \rightarrow 0} \frac{\arcsin x}{x} = \lim_{x \rightarrow 0} \frac{\left(\frac{1}{\sqrt{1-x^2}}\right)}{1} = 1.$$

### Problem Solving for Chapter 9

$$1. \text{ (a) } 1\left(\frac{1}{3}\right) + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) + \cdots = \sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^n$$

$$= \frac{1/3}{1 - (2/3)} = 1$$

$$\text{(b) } 0, \frac{1}{3}, \frac{2}{3}, 1, \text{ etc.}$$

$$\text{(c) } \lim_{n \rightarrow \infty} C_n = 1 - \sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^n = 1 - 1 = 0$$

$$2. \text{ Let } S = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\text{Then } \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$$= S + \frac{1}{2^2} + \frac{1}{4^2} + \cdots$$

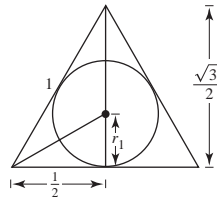
$$= S + \frac{1}{2^2} \left[ 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right] = S + \frac{1}{2^2} \left( \frac{\pi^2}{6} \right).$$

$$\text{Thus, } S = \frac{\pi^2}{6} - \frac{1}{4} \frac{\pi^2}{6} = \frac{\pi^2}{6} \left( \frac{3}{4} \right) = \frac{\pi^2}{8}.$$

$$3. \text{ If there are } n \text{ rows, then } a_n = \frac{n(n+1)}{2}.$$

For one circle,

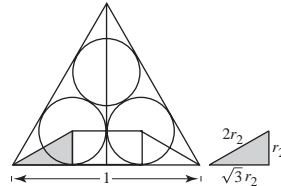
$$a_1 = 1 \text{ and } r_1 = \frac{1}{3} \left( \frac{\sqrt{3}}{2} \right) = \frac{\sqrt{3}}{6} = \frac{1}{2\sqrt{3}}.$$



For three circles,

$$a_2 = 3 \text{ and } 1 = 2\sqrt{3}r_2 + 2r_2$$

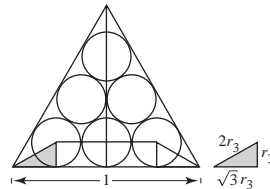
$$r_2 = \frac{1}{2 + 2\sqrt{3}}.$$



For six circles,

$$a_3 = 6 \text{ and } 1 = 2\sqrt{3}r_3 + 4r_3$$

$$r_3 = \frac{1}{2\sqrt{3} + 4}.$$



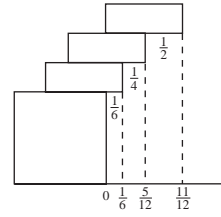
$$\text{Continuing this pattern, } r_n = \frac{1}{2\sqrt{3} + 2(n-1)}.$$

$$\text{Total Area} = (\pi r_n^2) a_n = \pi \left( \frac{1}{2\sqrt{3} + 2(n-1)} \right)^2 \frac{n(n+1)}{2}$$

$$A_n = \frac{\pi}{2} \frac{n(n+1)}{[2\sqrt{3} + 2(n-1)]^2}$$

$$\lim_{n \rightarrow \infty} A_n = \frac{\pi}{2} \cdot \frac{1}{4} = \frac{\pi}{8}$$

4. (a) Position the three blocks as indicated in the figure. The bottom block extends  $1/6$  over the edge of the table, the middle block extends  $1/4$  over the edge of the bottom block, and the top block extends  $1/2$  over the edge of the middle block.



The centers of gravity are located at

$$\text{bottom block: } \frac{1}{6} - \frac{1}{2} = -\frac{1}{3}$$

$$\text{middle block: } \frac{1}{6} + \frac{1}{4} - \frac{1}{2} = -\frac{1}{12}$$

$$\text{top block: } \frac{1}{6} + \frac{1}{4} + \frac{1}{2} - \frac{1}{2} = \frac{5}{12}$$

The center of gravity of the top 2 blocks is

$$\left(-\frac{1}{12} + \frac{5}{12}\right)/2 = \frac{1}{6},$$

which lies over the bottom block. The center of gravity of the 3 blocks is

$$\left(-\frac{1}{3} - \frac{1}{12} + \frac{5}{12}\right)/3 = 0$$

which lies over the table. Hence, the far edge of the top block lies

$$\frac{1}{6} + \frac{1}{4} + \frac{1}{2} = \frac{11}{12}$$

beyond the edge of the table.

- (b) Yes. If there are  $n$  blocks, then the edge of the top block lies  $\sum_{i=1}^n \frac{1}{2i}$  from the edge of the table. Using 4 blocks,

$$\sum_{i=1}^4 \frac{1}{2i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} = \frac{25}{24}$$

which shows that the top block extends beyond the table.

- (c) The blocks can extend any distance beyond the table because the series diverges:

$$\sum_{i=1}^{\infty} \frac{1}{2i} = \frac{1}{2} \sum_{i=1}^{\infty} \frac{1}{i} = \infty.$$

5. (a)  $\sum a_n x^n = 1 + 2x + 3x^2 + x^3 + 2x^4 + 3x^5 + \dots$

$$= (1 + x^3 + x^6 + \dots) + 2(x + x^4 + x^7 + \dots) + 3(x^2 + x^5 + x^8 + \dots)$$

$$= (1 + x^3 + x^6 + \dots)[1 + 2x + 3x^2]$$

$$= (1 + 2x + 3x^2) \frac{1}{1 - x^3}$$

$R = 1$  because each series in the second line has  $R = 1$ .

- (b)  $\sum a_n x^n = (a_0 + a_1 x + \dots + a_{p-1} x^{p-1}) + (a_0 x^p + a_1 x^{p+1} + \dots) + \dots$

$$= a_0(1 + x^p + \dots) + a_1 x(1 + x^p + \dots) + \dots + a_{p-1} x^{p-1}(1 + x^p + \dots)$$

$$= (a_0 + a_1 x + \dots + a_{p-1} x^{p-1})(1 + x^p + \dots)$$

$$= (a_0 + a_1 x + \dots + a_{p-1} x^{p-1}) \frac{1}{1 - x^p}$$

$$R = 1$$

(Assume all  $a_n > 0$ .)

$$6. a - \frac{b}{2} + \frac{a}{3} - \frac{b}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(a+b) + (a-b)}{2n}$$

$$\text{If } a = b, \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2a)}{2n} = a \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges conditionally.}$$

$$\text{If } a \neq b, \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(a+b)}{2n} + \sum_{n=1}^{\infty} \frac{a-b}{2n} \text{ diverges.}$$

No values of  $a$  and  $b$  give absolute convergence.  $a = b$  implies conditional convergence.

$$7. (a) \quad e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

$$\int xe^x dx = xe^x - e^x + C = \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!}$$

Letting  $x = 0$ , you have  $C = 1$ . Letting  $x = 1$ ,

$$e - e + 1 = \sum_{n=0}^{\infty} \frac{1}{(n+2)n!} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{(n+2)n!}$$

$$\text{Thus, } \sum_{n=1}^{\infty} \frac{1}{(n+2)n!} = \frac{1}{2}.$$

(b) Differentiating,

$$xe^x + e^x = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}.$$

Letting  $x = 1$ ,

$$2e = \sum_{n=0}^{\infty} \frac{n+1}{n!} \approx 5.4366.$$

$$9. \text{ Let } a_1 = \int_0^{\pi} \frac{\sin x}{x} dx, a_2 = -\int_{\pi}^{2\pi} \frac{\sin x}{x} dx, a_3 = \int_{2\pi}^{3\pi} \frac{\sin x}{x} dx, \text{ etc.}$$

Then,

$$\int_0^{\infty} \frac{\sin x}{x} dx = a_1 - a_2 + a_3 - a_4 + \cdots$$

Since  $\lim_{n \rightarrow \infty} a_n = 0$  and  $a_{n+1} < a_n$ , this series converges.

$$10. (a) \text{ If } p = 1, \int_2^{\infty} \frac{1}{x \ln x} dx = \ln \ln x \Big|_2^{\infty} \text{ diverges.}$$

$$\text{If } p > 1, \int_2^{\infty} \frac{1}{x(\ln x)^p} dx = \lim_{b \rightarrow \infty} \left[ \frac{(\ln b)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p} \right] \text{ converges.}$$

If  $p < 1$ , diverges.

$$(b) \sum_{n=4}^{\infty} \frac{1}{n \ln(n^2)} = \frac{1}{2} \sum_{n=4}^{\infty} \frac{1}{n \ln n} \text{ diverges by part (a).}$$

$$8. \quad e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \cdots + \frac{x^{12}}{6!} + \cdots$$

$$\frac{f^{(12)}(0)}{12!} = \frac{1}{6!} \Rightarrow f^{(12)}(0) = \frac{12!}{6!} = 665,280$$

11. (a)  $a_1 = 3.0$

$$a_2 \approx 1.73205$$

$$a_3 \approx 2.17533$$

$$a_4 \approx 2.27493$$

$$a_5 \approx 2.29672$$

$$a_6 \approx 2.30146$$

$$\lim_{n \rightarrow \infty} a_n = \frac{1 + \sqrt{13}}{2} \text{ [See part (b) for proof.]}$$

(b) Use mathematical induction to show the sequence is increasing. Clearly,  $a_2 = \sqrt{a + a_1} = \sqrt{a\sqrt{a}} > \sqrt{a} = a_1$ .

Now assume  $a_n > a_{n-1}$ . Then

$$\begin{aligned} a_n + a &> a_{n-1} + a \\ \sqrt{a_n + a} &> \sqrt{a_{n-1} + a} \\ a_{n+1} &> a_n. \end{aligned}$$

Use mathematical induction to show that the sequence is bounded above by  $a$ . Clearly,  $a_1 = \sqrt{a} < a$ .

Now assume  $a_n < a$ . Then  $a > a_n$  and  $a - 1 > 1$  implies

$$\begin{aligned} a(a - 1) &> a_n(1) \\ a^2 - a &> a_n \\ a^2 &> a_n + a \\ a &> \sqrt{a_n + a} = a_{n+1}. \end{aligned}$$

Hence, the sequence converges to some number  $L$ . To find  $L$ , assume  $a_{n+1} \approx a_n \approx L$ :

$$\begin{aligned} L &= \sqrt{a + L} \Rightarrow L^2 = a + L \Rightarrow L^2 - L - a = 0 \\ L &= \frac{1 \pm \sqrt{1 + 4a}}{2}. \end{aligned}$$

$$\text{Hence, } L = \frac{1 + \sqrt{1 + 4a}}{2}.$$

12. Let  $b_n = a_n r^n$ .

$$(b_n)^{1/n} = (a_n r^n)^{1/n} = a_n^{1/n} \cdot r \rightarrow Lr \text{ as } n \rightarrow \infty.$$

$$Lr < \frac{1}{r} = 1.$$

By the Root Test,  $\sum b_n$  converges  $\Rightarrow \sum a_n r^n$  converges.

13. (a)  $\sum_{n=1}^{\infty} \frac{1}{2^{n+(-1)^n}} = \frac{1}{2^{1-1}} + \frac{1}{2^{2+1}} + \frac{1}{2^{3-1}} + \frac{1}{2^{4+1}} + \frac{1}{2^{5-1}} + \dots$

$$S_1 = \frac{1}{2^0} = 1$$

$$S_2 = 1 + \frac{1}{8} = \frac{9}{8}$$

$$S_3 = \frac{9}{8} + \frac{1}{4} = \frac{11}{8}$$

$$S_4 = \frac{11}{8} + \frac{1}{32} = \frac{45}{32}$$

$$S_5 = \frac{45}{32} + \frac{1}{16} = \frac{47}{32}$$

## 13. —CONTINUED—

$$(b) \frac{a_{n+1}}{a_n} = \frac{2^{n+(-1)^n}}{2^{(n+1)+(-1)^{n+1}}} = \frac{2^{(-1)^n}}{2^{1+(-1)^{n+1}}}$$

This sequence is  $\frac{1}{8}, 2, \frac{1}{8}, 2, \dots$  which diverges.

$$(c) \sqrt[n]{\frac{1}{2^{n+(-1)^n}}} = \left(\frac{1}{2^n \cdot 2^{(-1)^n}}\right)^{1/n} \\ = \frac{1}{2 \cdot \sqrt[n]{2^{(-1)^n}}} \rightarrow \frac{1}{2} < 1 \text{ converges because } \{2^{(-1)^n}\} = \frac{1}{2}, 2, \frac{1}{2}, 2, \dots \text{ and } \sqrt[n]{1/2} \rightarrow 1 \text{ and } \sqrt[n]{2} \rightarrow 1.$$

$$14. (a) \frac{1}{0.99} = \frac{1}{1-0.01} = \sum_{n=0}^{\infty} (0.01)^n \\ = 1 + 0.01 + (0.01)^2 + \dots \\ = 1.010101 \dots$$

$$(b) \frac{1}{0.98} = \frac{1}{1-0.02} = \sum_{n=0}^{\infty} (0.02)^n \\ = 1 + 0.02 + (0.02)^2 + \dots \\ = 1 + 0.02 + 0.0004 + \dots \\ = 1.0204081632 \dots$$

$$15. S_6 = 130 + 70 + 40 = 240$$

$$S_7 = 240 + 130 + 70 = 440$$

$$S_8 = 440 + 240 + 130 = 810$$

$$S_9 = 810 + 440 + 240 = 1490$$

$$S_{10} = 1490 + 810 + 440 = 2740$$

$$16. (a) \text{Height} = 2 \left[ 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots \right] \\ = 2 \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \infty \left( p\text{-series, } p = \frac{1}{2} < 1 \right)$$

$$(b) S = 4\pi \left[ 1 + \frac{1}{2} + \frac{1}{3} + \dots \right] = 4\pi \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

$$(c) W = \frac{4}{3}\pi \left[ 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \dots \right] \\ = \frac{4}{3}\pi \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \text{ converges.}$$

$$17. (a) \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges (harmonic)}$$

(b) Let  $f(x) = \sin x$ . By the Mean Value Theorem,

$$|f(x) - f(y)| = f'(c)|x - y| = \cos(c)|x - y| \leq |x - y|,$$

where  $c$  is between  $x$  and  $y$ . Thus,

$$0 \leq \left| \sin\left(\frac{1}{2n}\right) - \sin\left(\frac{1}{2n+1}\right) \right| \\ \leq \left| \frac{1}{2n} - \frac{1}{2n+1} \right| \\ = \frac{1}{2n(2n+1)}$$

Because  $\sum_{n=1}^{\infty} \frac{1}{2n(2n+1)}$  converges, the Comparison Theorem

tells us that  $\sum_{n=1}^{\infty} \left[ \sin\left(\frac{1}{2n}\right) - \sin\left(\frac{1}{2n+1}\right) \right]$  converges.