CHAPTER 4
Integration

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Section 4.1 Antiderivatives and Indefinite Integration

1. $\frac{d}{dx}\left(\frac{3}{x^3} + C\right) = \frac{d}{dx}(3x^{-3} + C) = -9x^{-4} = \frac{-9}{x^4}$

2. $\frac{d}{dx}\left(x^4 + \frac{1}{x} + C\right) = 4x^3 - \frac{1}{x^2}$

3. $\frac{d}{dx}\left(\frac{1}{3}x^3 - 4x + C\right) = x^2 - 4 = (x - 2)(x + 2)$

4. $\frac{d}{dx}\left(\frac{2(x^2 + 3)}{3\sqrt{x}} + C\right) = \frac{d}{dx}\left(\frac{2}{3}x^{3/2} + 2x^{-1/2} + C\right)$

$= x^{1/2} - x^{-3/2} = \frac{x^2 - 1}{x^{3/2}}$

5. $\frac{dy}{dt} = 3t^2$

$y = t^3 + C$

Check: $\frac{d}{dt}[t^3 + C] = 3t^2$

6. $\frac{dr}{d\theta} = \pi$

$r = \pi\theta + C$

Check: $\frac{d}{d\theta}[\pi\theta + C] = \pi$

7. $\frac{dy}{dx} = x^{3/2}$

$y = \frac{2}{5}x^{5/2} + C$

Check: $\frac{d}{dx}\left[\frac{2}{5}x^{5/2} + C\right] = x^{3/2}$

8. $\frac{dy}{dx} = 2x^{-3}$

$y = \frac{2x^{-2}}{-2} + C = \frac{-1}{x^2} + C$

Check: $\frac{d}{dx}\left[\frac{-1}{x^2} + C\right] = 2x^{-3}$

	Given	Rewrite	Integrate	Simplify
9.	$\int \sqrt[3]{x}\, dx$	$\int x^{1/3}\, dx$	$\frac{x^{4/3}}{4/3} + C$	$\frac{3}{4}x^{4/3} + C$
10.	$\int \frac{1}{x^2}\, dx$	$\int x^{-2}\, dx$	$\frac{x^{-1}}{-1} + C$	$-\frac{1}{x} + C$
11.	$\int \frac{1}{x\sqrt{x}}\, dx$	$\int x^{-3/2}\, dx$	$\frac{x^{-1/2}}{-1/2} + C$	$-\frac{2}{\sqrt{x}} + C$
12.	$\int x(x^2 + 3)\, dx$	$\int (x^3 + 3x)\, dx$	$\frac{x^4}{4} + 3\left(\frac{x^2}{2}\right) + C$	$\frac{1}{4}x^4 + \frac{3}{2}x^2 + C$
13.	$\int \frac{1}{2x^3}\, dx$	$\frac{1}{2}\int x^{-3}\, dx$	$\frac{1}{2}\left(\frac{x^{-2}}{-2}\right) + C$	$-\frac{1}{4x^2} + C$
14.	$\int \frac{1}{(3x)^2}\, dx$	$\frac{1}{9}\int x^{-2}\, dx$	$\frac{1}{9}\left(\frac{x^{-1}}{-1}\right) + C$	$\frac{-1}{9x} + C$

15. $\int (x + 3)\,dx = \frac{x^2}{2} + 3x + C$

Check: $\frac{d}{dx}\left[\frac{x^2}{2} + 3x + C\right] = x + 3$

16. $\int (5 - x)\,dx = 5x - \frac{x^2}{2} + C$

Check: $\frac{d}{dx}\left[5x - \frac{x^2}{2} + C\right] = 5 - x$

17. $\int (2x - 3x^2)\,dx = x^2 - x^3 + C$

Check: $\frac{d}{dx}[x^2 - x^3 + C] = 2x - 3x^2$

18. $\int (4x^3 + 6x^2 - 1)\,dx = x^4 + 2x^3 - x + C$

Check: $\frac{d}{dx}[x^4 + 2x^3 - x + C] = 4x^3 + 6x^2 - 1$

19. $\int (x^3 + 2)\,dx = \frac{1}{4}x^4 + 2x + C$

Check: $\frac{d}{dx}\left(\frac{1}{4}x^4 + 2x + C\right) = x^3 + 2$

20. $\int (x^3 - 4x + 2)\,dx = \frac{x^4}{4} - 2x^2 + 2x + C$

Check: $\frac{d}{dx}\left[\frac{x^4}{4} - 2x^2 + 2x + C\right] = x^3 - 4x + 2$

21. $\int (x^{3/2} + 2x + 1)\,dx = \frac{2}{5}x^{5/2} + x^2 + x + C$

Check: $\frac{d}{dx}\left(\frac{2}{5}x^{5/2} + x^2 + x + C\right) = x^{3/2} + 2x + 1$

22. $\int\left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right)dx = \int\left(x^{1/2} + \frac{1}{2}x^{-1/2}\right)dx$

$= \frac{x^{3/2}}{3/2} + \frac{1}{2}\left(\frac{x^{1/2}}{1/2}\right) + C$

$= \frac{2}{3}x^{3/2} + x^{1/2} + C$

Check: $\frac{d}{dx}\left(\frac{2}{3}x^{3/2} + x^{1/2} + C\right) = x^{1/2} + \frac{1}{2}x^{-1/2}$

$= \sqrt{x} + \frac{1}{2\sqrt{x}}$

23. $\int \sqrt[3]{x^2}\,dx = \int x^{2/3}\,dx = \frac{x^{5/3}}{5/3} + C = \frac{3}{5}x^{5/3} + C$

Check: $\frac{d}{dx}\left(\frac{3}{5}x^{5/3} + C\right) = x^{2/3} = \sqrt[3]{x^2}$

24. $\int\left(\sqrt[4]{x^3} + 1\right)dx = \int (x^{3/4} + 1)\,dx = \frac{4}{7}x^{7/4} + x + C$

Check: $\frac{d}{dx}\left(\frac{4}{7}x^{7/4} + x + C\right) = x^{3/4} + 1 = \sqrt[4]{x^3} + 1$

25. $\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$

Check: $\frac{d}{dx}\left(-\frac{1}{2x^2} + C\right) = \frac{1}{x^3}$

26. $\int \frac{1}{x^4}\,dx = \int x^{-4}\,dx = \frac{x^{-3}}{-3} + C = -\frac{1}{3x^3} + C$

Check: $\frac{d}{dx}\left(-\frac{1}{3x^3} + C\right) = \frac{1}{x^4}$

27. $\int \frac{x^2 + x + 1}{\sqrt{x}}\,dx = \int (x^{3/2} + x^{1/2} + x^{-1/2})\,dx = \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2x^{1/2} + C = \frac{2}{15}x^{1/2}(3x^2 + 5x + 15) + C$

Check: $\frac{d}{dx}\left(\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2x^{1/2} + C\right) = x^{3/2} + x^{1/2} + x^{-1/2} = \frac{x^2 + x + 1}{\sqrt{x}}$

28. $\int \frac{x^2 + 2x - 3}{x^4}\,dx = \int (x^{-2} + 2x^{-3} - 3x^{-4})\,dx = \frac{x^{-1}}{-1} + \frac{2x^{-2}}{-2} - \frac{3x^{-3}}{-3} + C = \frac{-1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + C$

Check: $\frac{d}{dx}\left[-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + C\right] = x^{-2} + 2x^{-3} - 3x^{-4} = \frac{x^2 + 2x - 3}{x^4}$

29. $\int (x+1)(3x-2)\,dx = \int (3x^2+x-2)\,dx$

$= x^3 + \frac{1}{2}x^2 - 2x + C$

Check: $\frac{d}{dx}\left(x^3 + \frac{1}{2}x^2 - 2x + C\right) = 3x^2 + x - 2$

$= (x+1)(3x-2)$

30. $\int (2t^2-1)^2\,dt = \int (4t^4 - 4t^2 + 1)\,dt$

$= \frac{4}{5}t^5 - \frac{4}{3}t^3 + t + C$

Check: $\frac{d}{dt}\left(\frac{4}{5}t^5 - \frac{4}{3}t^3 + t + C\right) = 4t^4 - 4t^2 + 1$

$= (2t^2-1)^2$

31. $\int y^2\sqrt{y}\,dy = \int y^{5/2}\,dy = \frac{2}{7}y^{7/2} + C$

Check: $\frac{d}{dy}\left(\frac{2}{7}y^{7/2} + C\right) = y^{5/2} = y^2\sqrt{y}$

32. $\int (1+3t)t^2\,dt = \int (t^2 + 3t^3)\,dt = \frac{1}{3}t^3 + \frac{3}{4}t^4 + C$

Check: $\frac{d}{dt}\left(\frac{1}{3}t^3 + \frac{3}{4}t^4 + C\right) = t^2 + 3t^3 = (1+3t)t^2$

33. $\int dx = \int 1\,dx = x + C$

Check: $\frac{d}{dx}(x + C) = 1$

34. $\int 3\,dt = 3t + C$

Check: $\frac{d}{dt}(3t + C) = 3$

35. $\int (2\sin x + 3\cos x)\,dx = -2\cos x + 3\sin x + C$

Check: $\frac{d}{dx}(-2\cos x + 3\sin x + C) = 2\sin x + 3\cos x$

36. $\int (t^2 - \sin t)\,dt = \frac{1}{3}t^3 + \cos t + C$

Check: $\frac{d}{dt}\left(\frac{1}{3}t^3 + \cos t + C\right) = t^2 - \sin t$

37. $\int (1 - \csc t\cot t)\,dt = t + \csc t + C$

Check: $\frac{d}{dt}(t + \csc t + C) = 1 - \csc t\cot t$

38. $\int (\theta^2 + \sec^2\theta)\,d\theta = \frac{1}{3}\theta^3 + \tan\theta + C$

Check: $\frac{d}{d\theta}\left(\frac{1}{3}\theta^3 + \tan\theta + C\right) = \theta^2 + \sec^2\theta$

39. $\int (\sec^2\theta - \sin\theta)\,d\theta = \tan\theta + \cos\theta + C$

Check: $\frac{d}{d\theta}(\tan\theta + \cos\theta + C) = \sec^2\theta - \sin\theta$

40. $\int \sec y(\tan y - \sec y)\,dy = \int (\sec y\tan y - \sec^2 y)\,dy$

$= \sec y - \tan y + C$

Check: $\frac{d}{dy}(\sec y - \tan y + C) = \sec y\tan y - \sec^2 y$

$= \sec y(\tan y - \sec y)$

41. $\int (\tan^2 y + 1)\,dy = \int \sec^2 y\,dy = \tan y + C$

Check: $\frac{d}{dy}(\tan y + C) = \sec^2 y = \tan^2 y + 1$

42. $\int \frac{\cos x}{1-\cos^2 x}\,dx = \int \frac{\cos x}{\sin^2 x}\,dx = \int \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)dx$

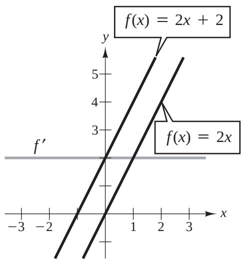
$= \int \csc x\cot x\,dx = -\csc x + C$

Check: $\frac{d}{dx}[-\csc x + C] = \csc x\cot x + \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}$

$= \frac{\cos x}{1-\cos^2 x}$

43. $f'(x) = 2$

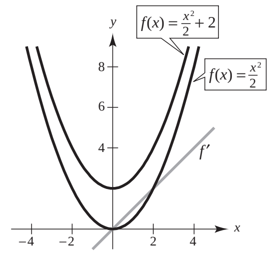
$f(x) = 2x + C$



Answers will vary.

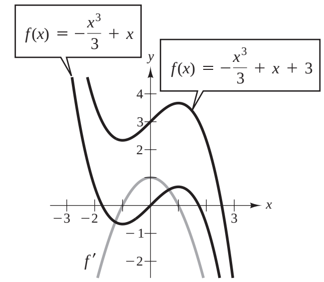
44. $f'(x) = x$

$f(x) = \dfrac{x^2}{2} + C$



45. $f'(x) = 1 - x^2$

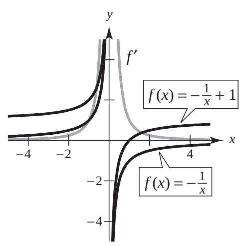
$f(x) = x - \dfrac{x^3}{3} + C$



Answers will vary.

46. $f'(x) = \dfrac{1}{x^2}$

$f(x) = -\dfrac{1}{x} + C$



47. $\dfrac{dy}{dx} = 2x - 1,\ (1, 1)$

$y = \displaystyle\int (2x - 1)\,dx = x^2 - x + C$

$1 = (1)^2 - (1) + C \implies C = 1$

$y = x^2 - x + 1$

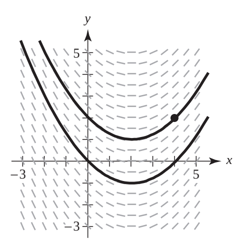
48. $\dfrac{dy}{dx} = 2(x - 1) = 2x - 2,\ (3, 2)$

$y = \displaystyle\int 2(x - 1)\,dx = x^2 - 2x + C$

$2 = (3)^2 - 2(3) + C \implies C = -1$

$y = x^2 - 2x - 1$

49. (a) Answers will vary.



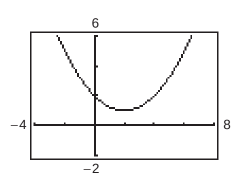
(b) $\dfrac{dy}{dx} = \dfrac{1}{2}x - 1,\ (4, 2)$

$y = \dfrac{x^2}{4} - x + C$

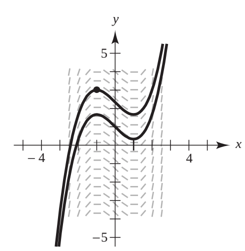
$2 = \dfrac{4^2}{4} - 4 + C$

$2 = C$

$y = \dfrac{x^2}{4} - x + 2$



50. (a)



(b) $\dfrac{dy}{dx} = x^2 - 1, \quad (-1, 3)$

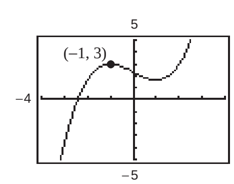
$y = \dfrac{x^3}{3} - x + C$

$3 = \dfrac{(-1)^3}{3} - (-1) + C$

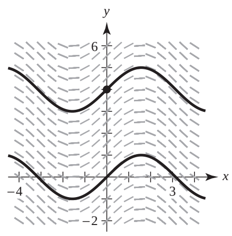
$3 = -\dfrac{1}{3} + 1 + C$

$C = \dfrac{7}{3}$

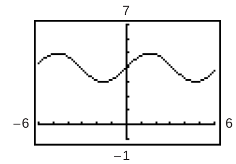
$y = \dfrac{x^3}{3} - x + \dfrac{7}{3}$



51. (a)



(b) $\dfrac{dy}{dx} = \cos x, \quad (0, 4)$

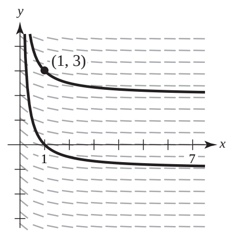


$$y = \int \cos x\, dx = \sin x + C$$

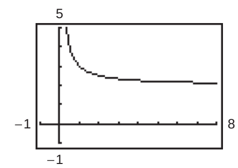
$$4 = \sin(0) + C \Rightarrow C = 4$$

$$y = \sin x + 4$$

52. (a)



(b) $\dfrac{dy}{dx} = \dfrac{-1}{x^2}, \; x > 0, \quad (1, 3)$

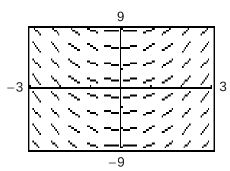


$$y = \int -\frac{1}{x^2}\, dx = \int -x^{-2}\, dx = \frac{-x^{-1}}{-1} + C = \frac{1}{x} + C$$

$$3 = \frac{1}{1} + C \Rightarrow C = 2$$

$$y = \frac{1}{x} + 2$$

53. (a)



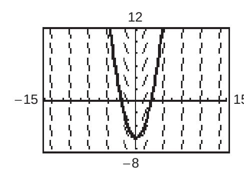
(b) $\dfrac{dy}{dx} = 2x, \quad (-2, -2)$

$$y = \int 2x\, dx = x^2 + C$$

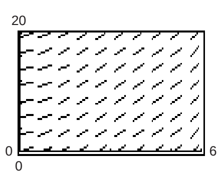
$$-2 = (-2)^2 + C = 4 + C \Rightarrow C = -6$$

$$y = x^2 - 6$$

(c)



54. (a)



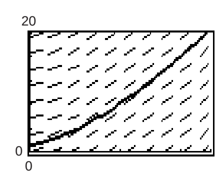
(b) $\dfrac{dy}{dx} = 2\sqrt{x}, \quad (4, 12)$

$$y = \int 2x^{1/2}\, dx = \frac{4}{3}x^{3/2} + C$$

$$12 = \frac{4}{3}(4)^{3/2} + C = \frac{4}{3}(8) + C = \frac{32}{3} + C \Rightarrow C = \frac{4}{3}$$

$$y = \frac{4}{3}x^{3/2} + \frac{4}{3}$$

(c)



55. $f'(x) = 4x, f(0) = 6$

$$f(x) = \int 4x\, dx = 2x^2 + C$$

$$f(0) = 6 = 2(0)^2 + C \Rightarrow C = 6$$

$$f(x) = 2x^2 + 6$$

56. $g'(x) = 6x^2, g(0) = -1$

$$g(x) = \int 6x^2\, dx = 2x^3 + C$$

$$g(0) = -1 = 2(0)^3 + C \Rightarrow C = -1$$

$$g(x) = 2x^3 - 1$$

57. $h'(t) = 8t^3 + 5, h(1) = -4$

$$h(t) = \int (8t^3 + 5)\, dt = 2t^4 + 5t + C$$

$$h(1) = -4 = 2 + 5 + C \Rightarrow C = -11$$

$$h(t) = 2t^4 + 5t - 11$$

58. $f'(s) = 6s - 8s^3, \; f(2) = 3$

$$f(s) = \int (6s - 8s^3)\, ds = 3s^2 - 2s^4 + C$$

$$f(2) = 3 = 3(2)^2 - 2(2)^4 + C = 12 - 32 + C \Rightarrow C = 23$$

$$f(s) = 3s^2 - 2s^4 + 23$$

59. $f''(x) = 2$

$f'(2) = 5$

$f(2) = 10$

$f'(x) = \int 2\,dx = 2x + C_1$

$f'(2) = 4 + C_1 = 5 \implies C_1 = 1$

$f'(x) = 2x + 1$

$f(x) = \int (2x + 1)\,dx = x^2 + x + C_2$

$f(2) = 6 + C_2 = 10 \implies C_2 = 4$

$f(x) = x^2 + x + 4$

60. $f''(x) = x^2$

$f'(0) = 6$

$f(0) = 3$

$f'(x) = \int x^2\,dx = \frac{1}{3}x^3 + C_1$

$f'(0) = 0 + C_1 = 6 \implies C_1 = 6$

$f'(x) = \frac{1}{3}x^3 + 6$

$f(x) = \int \left(\frac{1}{3}x^3 + 6\right) dx = \frac{1}{12}x^4 + 6x + C_2$

$f(0) = 0 + 0 + C_2 = 3 \implies C_2 = 3$

$f(x) = \frac{1}{12}x^4 + 6x + 3$

61. $f''(x) = x^{-3/2}$

$f'(4) = 2$

$f(0) = 0$

$f'(x) = \int x^{-3/2}\,dx = -2x^{-1/2} + C_1 = -\frac{2}{\sqrt{x}} + C_1$

$f'(4) = -\frac{2}{2} + C_1 = 2 \implies C_1 = 3$

$f'(x) = -\frac{2}{\sqrt{x}} + 3$

$f(x) = \int (-2x^{-1/2} + 3)\,dx = -4x^{1/2} + 3x + C_2$

$f(0) = 0 + 0 + C_2 = 0 \implies C_2 = 0$

$f(x) = -4x^{1/2} + 3x = -4\sqrt{x} + 3x$

62. $f''(x) = \sin x$

$f'(0) = 1$

$f(0) = 6$

$f'(x) = \int \sin x\,dx = -\cos x + C_1$

$f'(0) = -1 + C_1 = 1 \implies C_1 = 2$

$f'(x) = -\cos x + 2$

$f(x) = \int (-\cos x + 2)\,dx = -\sin x + 2x + C_2$

$f(0) = 0 + 0 + C_2 = 6 \implies C_2 = 6$

$f(x) = -\sin x + 2x + 6$

63. (a) $h(t) = \int (1.5t + 5)\,dt = 0.75t^2 + 5t + C$

$h(0) = 0 + 0 + C = 12 \implies C = 12$

$h(t) = 0.75t^2 + 5t + 12$

(b) $h(6) = 0.75(6)^2 + 5(6) + 12 = 69$ cm

64. $\frac{dP}{dt} = k\sqrt{t},\ 0 \le t \le 10$

$P(t) = \int kt^{1/2}\,dt = \frac{2}{3}kt^{3/2} + C$

$P(0) = 0 + C = 500 \implies C = 500$

$P(1) = \frac{2}{3}k + 500 = 600 \implies k = 150$

$P(t) = \frac{2}{3}(150)t^{3/2} + 500 = 100t^{3/2} + 500$

$P(7) = 100(7)^{3/2} + 500 \approx 2352$ bacteria

65. $f(0) = -4$. Graph of f' is given.

(a) $f'(4) \approx -1.0$

(b) No. The slopes of the tangent lines are greater than 2 on $[0, 2]$. Therefore, f must increase more than 4 units on $[0, 4]$.

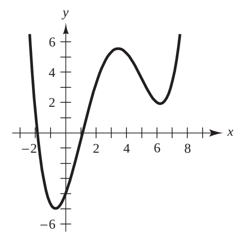
(c) No, $f(5) < f(4)$ because f is decreasing on $[4, 5]$.

(d) f is a maximum at $x = 3.5$ because $f'(3.5) \approx 0$ and the First Derivative Test.

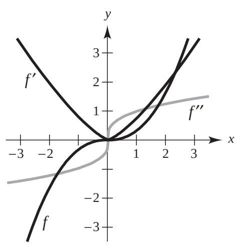
(e) f is concave upward when f' is increasing on $(-\infty, 1)$ and $(5, \infty)$. f is concave downward on $(1, 5)$. Points of inflection at $x = 1, 5$.

(f) f'' is a minimum at $x = 3$.

(g)



66. Since f'' is negative on $(-\infty, 0)$, f' is decreasing on $(-\infty, 0)$. Since f'' is positive on $(0, \infty)$, f' is increasing on $(0, \infty)$. f' has a relative minimum at $(0, 0)$. Since f' is positive on $(-\infty, \infty)$, f is increasing on $(-\infty, \infty)$.



67. $a(t) = -32 \text{ ft/sec}^2$

$v(t) = \int -32\,dt = -32t + C_1$

$v(0) = 60 = C_1$

$s(t) = \int (-32t + 60)\,dt = -16t^2 + 60t + C_2$

$s(0) = 6 = C_2$

$s(t) = -16t^2 + 60t + 6$, Position function

The ball reaches its maximum height when

$v(t) = -32t + 60 = 0$

$32t = 60$

$t = \frac{15}{8}$ seconds.

$s\left(\frac{15}{8}\right) = -16\left(\frac{15}{8}\right)^2 + 60\left(\frac{15}{8}\right) + 6 = 62.25$ feet

68. $f''(t) = a(t) = -32 \text{ ft/sec}^2$

$f'(0) = v_0$

$f(0) = s_0$

$f'(t) = v(t) = \int -32\,dt = -32t + C_1$

$f'(0) = 0 + C_1 = v_0 \implies C_1 = v_0$

$f'(t) = -32t + v_0$

$f(t) = s(t) = \int (-32t + v_0)\,dt = -16t^2 + v_0 t + C_2$

$f(0) = 0 + 0 + C_2 = s_0 \implies C_2 = s_0$

$f(t) = -16t^2 + v_0 t + s_0$

69. From Exercise 68, we have:

$s(t) = -16t^2 + v_0 t$

$s'(t) = -32t + v_0 = 0$ when $t = \dfrac{v_0}{32}$ = time to reach maximum height.

$$s\left(\frac{v_0}{32}\right) = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) = 550$$

$$-\frac{{v_0}^2}{64} + \frac{{v_0}^2}{32} = 550$$

$${v_0}^2 = 35{,}200$$

$$v_0 \approx 187.617 \text{ ft/sec}$$

70. $v_0 = 16$ ft/sec

$s_0 = 64$ ft

(a) $s(t) = -16t^2 + 16t + 64 = 0$

$-16(t^2 - t - 4) = 0$

$$t = \frac{1 \pm \sqrt{17}}{2}$$

Choosing the positive value,

$$t = \frac{1 + \sqrt{17}}{2} \approx 2.562 \text{ seconds.}$$

(b) $v(t) = s'(t) = -32t + 16$

$$v\left(\frac{1+\sqrt{17}}{2}\right) = -32\left(\frac{1+\sqrt{17}}{2}\right) + 16$$

$$= -16\sqrt{17} \approx -65.970 \text{ ft/sec}$$

71. $a(t) = -9.8$

$$v(t) = \int -9.8\, dt = -9.8t + C_1$$

$$v(0) = v_0 = C_1 \implies v(t) = -9.8t + v_0$$

$$f(t) = \int (-9.8t + v_0)\, dt = -4.9t^2 + v_0 t + C_2$$

$$f(0) = s_0 = C_2 \implies f(t) = -4.9t^2 + v_0 t + s_0$$

72. From Exercise 71, $f(t) = -4.9t^2 + 1800$. (Using the canyon floor as position 0.)

$$f(t) = 0 = -4.9t^2 + 1800$$

$$4.9t^2 = 1800$$

$$t^2 = \frac{1800}{4.9} \implies t \approx 9.2 \text{ sec}$$

73. From Exercise 71, $f(t) = -4.9t^2 + 10t + 2$.

$v(t) = -9.8t + 10 = 0$ (Maximum height when $v = 0$.)

$$9.8t = 10$$

$$t = \frac{10}{9.8}$$

$$f\left(\frac{10}{9.8}\right) \approx 7.1 \text{ m}$$

74. From Exercise 71, $f(t) = -4.9t^2 + v_0 t + 2$. If

$$f(t) = 200 = -4.9t^2 + v_0 t + 2,$$

then

$$v(t) = -9.8t + v_0 = 0$$

for this t value. Hence, $t = v_0/9.8$ and we solve

$$-4.9\left(\frac{v_0}{9.8}\right)^2 + v_0\left(\frac{v_0}{9.8}\right) + 2 = 200$$

$$\frac{-4.9\, v_0^2}{(9.8)^2} + \frac{v_0^2}{9.8} = 198$$

$$-4.9\, v_0^2 + 9.8\, v_0^2 = (9.8)^2\, 198$$

$$4.9\, v_0^2 = (9.8)^2\, 198$$

$$v_0^2 = 3880.8 \implies v_0 \approx 62.3 \text{ m/sec.}$$

75. $a = -1.6$

$v(t) = \int -1.6\, dt = -1.6t + v_0 = -1.6t$, since the stone was dropped, $v_0 = 0$.

$$s(t) = \int (-1.6t)\, dt = -0.8t^2 + s_0$$

$$s(20) = 0 \implies -0.8(20)^2 + s_0 = 0$$

$$s_0 = 320$$

Thus, the height of the cliff is 320 meters.

$$v(t) = -1.6t$$

$$v(20) = -32 \text{ m/sec}$$

76. $\int v\,dv = -GM\int \frac{1}{y^2}\,dy$

$$\frac{1}{2}v^2 = \frac{GM}{y} + C$$

When $y = R$, $v = v_0$.

$$\frac{1}{2}v_0^2 = \frac{GM}{R} + C$$

$$C = \frac{1}{2}v_0^2 - \frac{GM}{R}$$

$$\frac{1}{2}v^2 = \frac{GM}{y} + \frac{1}{2}v_0^2 - \frac{GM}{R}$$

$$v^2 = \frac{2GM}{y} + v_0^2 - \frac{2GM}{R}$$

$$v^2 = v_0^2 + 2GM\left(\frac{1}{y} - \frac{1}{R}\right)$$

77. $x(t) = t^3 - 6t^2 + 9t - 2 \qquad 0 \le t \le 5$

(a) $v(t) = x'(t) = 3t^2 - 12t + 9$

$= 3(t^2 - 4t + 3) = 3(t - 1)(t - 3)$

$a(t) = v'(t) = 6t - 12 = 6(t - 2)$

(b) $v(t) > 0$ when $0 < t < 1$ or $3 < t < 5$.

(c) $a(t) = 6(t - 2) = 0$ when $t = 2$.

$v(2) = 3(1)(-1) = -3$

78. $x(t) = (t - 1)(t - 3)^2 \qquad 0 \le t \le 5$

$= t^3 - 7t^2 + 15t - 9$

(a) $v(t) = x'(t) = 3t^2 - 14t + 15 = (3t - 5)(t - 3)$

$a(t) = v'(t) = 6t - 14$

(b) $v(t) > 0$ when $0 < t < \frac{5}{3}$ and $3 < t < 5$.

(c) $a(t) = 6t - 14 = 0$ when $t = \frac{7}{3}$.

$$v\left(\frac{7}{3}\right) = \left(3\left(\frac{7}{3}\right) - 5\right)\left(\frac{7}{3} - 3\right) = 2\left(-\frac{2}{3}\right) = -\frac{4}{3}$$

79. $v(t) = \frac{1}{\sqrt{t}} = t^{-1/2} \qquad t > 0$

$x(t) = \int v(t)\,dt = 2t^{1/2} + C$

$x(1) = 4 = 2(1) + C \implies C = 2$

$x(t) = 2t^{1/2} + 2$ position function

$a(t) = v'(t) = -\frac{1}{2}t^{-3/2} = \frac{-1}{2t^{3/2}}$ acceleration

80. (a) $a(t) = \cos t$

$v(t) = \int a(t)\,dt = \int \cos t\,dt = \sin t + C_1 = \sin t$ (since $v_0 = 0$)

$f(t) = \int v(t)\,dt = \int \sin t\,dt = -\cos t + C_2$

$f(0) = 3 = -\cos(0) + C_2 = -1 + C_2 \implies C_2 = 4$

$f(t) = -\cos t + 4$

(b) $v(t) = 0 = \sin t$ for $t = k\pi$, $k = 0, 1, 2, \ldots$

81. (a) $v(0) = 25 \text{ km/hr} = 25 \cdot \frac{1000}{3600} = \frac{250}{36} \text{ m/sec}$

$v(13) = 80 \text{ km/hr} = 80 \cdot \frac{1000}{3600} = \frac{800}{36} \text{ m/sec}$

$a(t) = a$ (constant acceleration)

$v(t) = at + C$

$v(0) = \frac{250}{36} \implies v(t) = at + \frac{250}{36}$

$v(13) = \frac{800}{36} = 13a + \frac{250}{36}$

$\frac{550}{36} = 13a$

$a = \frac{550}{468} = \frac{275}{234} \approx 1.175 \text{ m/sec}^2$

(b) $s(t) = a\frac{t^2}{2} + \frac{250}{36}t \quad (s(0) = 0)$

$s(13) = \frac{275}{234}\frac{(13)^2}{2} + \frac{250}{36}(13) \approx 189.58 \text{ m}$

82. $v(0) = 45 \text{ mph} = 66 \text{ ft/sec}$

$30 \text{ mph} = 44 \text{ ft/sec}$

$15 \text{ mph} = 22 \text{ ft/sec}$

$a(t) = -a$

$v(t) = -at + 66$

$s(t) = -\frac{a}{2}t^2 + 66t$ (Let $s(0) = 0$.)

$v(t) = 0$ after car moves 132 ft.

$-at + 66 = 0$ when $t = \frac{66}{a}$.

$s\left(\frac{66}{a}\right) = -\frac{a}{2}\left(\frac{66}{a}\right)^2 + 66\left(\frac{66}{a}\right)$

$= 132$ when $a = \frac{33}{2} = 16.5$.

$a(t) = -16.5$

$v(t) = -16.5t + 66$

$s(t) = -8.25t^2 + 66t$

(a) $-16.5t + 66 = 44$

$t = \frac{22}{16.5} \approx 1.333$

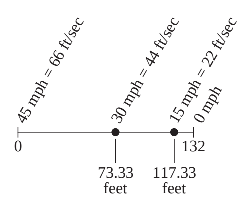
$s\left(\frac{22}{16.5}\right) \approx 73.33 \text{ ft}$

(b) $-16.5t + 66 = 22$

$t = \frac{44}{16.5} \approx 2.667$

$s\left(\frac{44}{16.5}\right) \approx 117.33 \text{ ft}$

(c)



It takes 1.333 seconds to reduce the speed from 45 mph to 30 mph, 1.333 seconds to reduce the speed from 30 mph to 15 mph, and 1.333 seconds to reduce the speed from 15 mph to 0 mph. Each time, less distance is needed to reach the next speed reduction.

83. Truck: $v(t) = 30$

$s(t) = 30t$ (Let $s(0) = 0$.)

Automobile: $a(t) = 6$

$v(t) = 6t$ (Let $v(0) = 0$.)

$s(t) = 3t^2$ (Let $s(0) = 0$.)

At the point where the automobile overtakes the truck:

$30t = 3t^2$

$0 = 3t^2 - 30t$

$0 = 3t(t - 10)$ when $t = 10$ sec.

(a) $s(10) = 3(10)^2 = 300 \text{ ft}$

(b) $v(10) = 6(10) = 60 \text{ ft/sec} \approx 41 \text{ mph}$

84. $\dfrac{(1 \text{ mi/hr})(5280 \text{ ft/mi})}{(3600 \text{ sec/hr})} = \dfrac{22}{15}$ ft/sec

(a)

t	0	5	10	15	20	25	30
V_1(ft/sec)	0	3.67	10.27	23.47	42.53	66	95.33
V_2(ft/sec)	0	30.8	55.73	74.8	88	93.87	95.33

(b) $V_1(t) = 0.1068t^2 - 0.0416t + 0.3679$

$V_2(t) = -0.1208t^2 + 6.7991t - 0.0707$

(c) $S_1(t) = \displaystyle\int V_1(t)\,dt = \frac{0.1068}{3}t^3 - \frac{0.0416}{2}t^2 + 0.3679t$

$S_2(t) = \displaystyle\int V_2(t)\,dt = -\frac{0.1208t^3}{3} + \frac{6.7991t^2}{2} - 0.0707t$

[In both cases, the constant of integration is 0 because $S_1(0) = S_2(0) = 0$.]

$S_1(30) \approx 953.5$ feet

$S_2(30) \approx 1970.3$ feet

The second car was going faster than the first until the end.

85. $a(t) = k$

$v(t) = kt$

$s(t) = \dfrac{k}{2}t^2$ since $v(0) = s(0) = 0$.

At the time of lift-off, $kt = 160$ and $(k/2)t^2 = 0.7$. Since $(k/2)t^2 = 0.7$,

$$t = \sqrt{\frac{1.4}{k}}$$

$$v\left(\sqrt{\frac{1.4}{k}}\right) = k\sqrt{\frac{1.4}{k}} = 160$$

$$1.4k = 160^2 \implies k = \frac{160^2}{1.4}$$

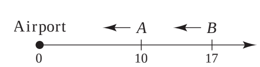
$$\approx 18{,}285.714 \text{ mi/hr}^2$$

$$\approx 7.45 \text{ ft/sec}^2.$$

86. Let the aircrafts be located 10 and 17 miles away from the airport, as indicated in the figure.

$v_A(t) = k_A t - 150$ $\qquad v_B = k_B t - 250$

$s_A(t) = \dfrac{1}{2}k_A t^2 - 150t + 10$ $\qquad s_B = \dfrac{1}{2}k_B t^2 - 250t + 17$



(a) When aircraft A lands at time t_A you have

$$v_A(t_A) = k_A t_A - 150 = -100 \implies k_A = \frac{50}{t_A}$$

$$s_A(t_A) = \frac{1}{2}k_A t_A^2 - 150t_A + 10 = 0$$

$$\frac{1}{2}\left(\frac{50}{t_A}\right)t_A^2 - 150t_A = -10$$

$$125t_A = 10$$

$$t_A = \frac{10}{125}.$$

—CONTINUED—

86. —CONTINUED—

$$k_A = \frac{50}{t_A} = 50\left(\frac{125}{10}\right) = 625 \Longrightarrow s_A(t) = \frac{625}{2}t^2 - 150t + 10$$

Similarly, when aircraft B lands at time t_B you have

$$v_B(t_B) = k_B t_B - 250 = -115 \Longrightarrow k_B = \frac{135}{t_B}$$

$$s_B(t_B) = \frac{1}{2}k_B t_B^2 - 250t_B + 17 = 0$$

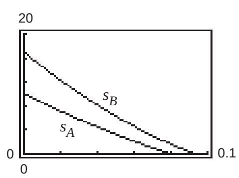
$$\frac{1}{2}\left(\frac{135}{t_B}\right)t_B^2 - 250t_B = -17$$

$$\frac{365}{2}t_B = 17$$

$$t_B = \frac{34}{365}\text{ hr}.$$

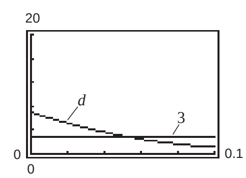
$$k_B = \frac{135}{t_B} = 135\left(\frac{365}{34}\right) = \frac{49{,}275}{34} \Longrightarrow s_B(t) = \frac{49{,}275}{68}t^2 - 250t + 17$$

(b)



(c) $d = s_B(t) - s_A(t) = \frac{28{,}025}{68}t^2 - 100t + 7$

Yes, $d < 3$ for $t > 0.0505$ hr.



87. True

88. True

89. True

90. True

91. False. For example, $\int x \cdot x\,dx \neq \int x\,dx \cdot \int x\,dx$ because $\frac{x^3}{3} + C \neq \left(\frac{x^2}{2} + C_1\right)\left(\frac{x^2}{2} + C_2\right)$.

92. False. f has an infinite number of antiderivatives, each differing by a constant.

93. $f''(x) = 2x$

$f'(x) = x^2 + C$

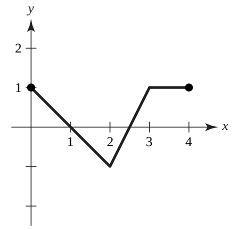
$f'(2) = 0 \Longrightarrow 4 + C = 0 \Longrightarrow C = -4$

$f(x) = \frac{x^3}{3} - 4x + C_1$

$f(2) = 0 \Longrightarrow \frac{8}{3} - 8 + C_1 = 0 \Longrightarrow C_1 = \frac{16}{3}$

Answer: $f(x) = \frac{x^3}{3} - 4x + \frac{16}{3}$

94. $f'(x) = \begin{cases} -1, & 0 \le x < 2 \\ 2\ , & 2 < x < 3 \\ 0\ , & 3 < x \le 4 \end{cases}$



$f(x) = \begin{cases} -x + C_1, & 0 \le x < 2 \\ 2x + C_2\ , & 2 < x < 3 \\ C_3\ , & 3 < x \le 4 \end{cases}$

$f(0) = 1 \implies C_1 = 1$

f continuous at $x = 2 \implies -2 + 1 = 4 + C_2 \implies C_2 = -5$

f continuous at $x = 3 \implies 6 - 5 = C_3 = 1$

$f(x) = \begin{cases} -x + 1, & 0 \le x < 2 \\ 2x - 5\ , & 2 \le x < 3 \\ 1\ , & 3 \le x \le 4 \end{cases}$

95. $f'(x) = \begin{cases} 1, & 0 \le x < 2 \\ 3x, & 2 \le x \le 5 \end{cases}$

$f(x) = \begin{cases} x + C_1, & 0 \le x < 2 \\ \dfrac{3x^2}{2} + C_2, & 2 \le x \le 5 \end{cases}$

$f(1) = 3 \implies 1 + C_1 = 3 \implies C_1 = 2$

f is continuous: Values must agree at $x = 2$:

$4 = 6 + C_2 \implies C_2 = -2$

$f(x) = \begin{cases} x + 2, & 0 \le x < 2 \\ \dfrac{3x^2}{2} - 2, & 2 \le x \le 5 \end{cases}$

The left and right hand derivatives at $x = 2$ do not agree. Hence f is not differentiable at $x = 2$.

96. $\dfrac{d}{dx}\left[[s(x)]^2 + [c(x)]^2\right] = 2s(x)s'(x) + 2c(x)c'(x)$

$= 2s(x)c(x) - 2c(x)s(x)$

$= 0$

Thus, $[s(x)]^2 + [c(x)]^2 = k$ for some constant k. Since, $s(0) = 0$ and $c(0) = 1$, $k = 1$.

Therefore, $[s(x)]^2 + [c(x)]^2 = 1$.

[Note that $s(x) = \sin x$ and $c(x) = \cos x$ satisfy these properties.]

97. $f(x + y) = f(x)f(y) - g(x)g(y)$

$g(x + y) = f(x)g(y) + g(x)f(y)$

$f'(0) = 0$

[Note: $f(x) = \cos x$ and $g(x) = \sin x$ satisfy these conditions]

$f'(x + y) = f(x)f'(y) - g(x)g'(y)$ (Differentiate with respect to y)

$g'(x + y) = f(x)g'(y) + g(x)f'(y)$ (Differentiate with respect to y)

Letting $y = 0$, $f'(x) = f(x)f'(0) - g(x)g'(0) = -g(x)g'(0)$

$g'(x) = f(x)g'(0) + g(x)f'(0) = f(x)g'(0)$

—CONTINUED—

97. —CONTINUED—

Hence, $2f(x)f'(x) = -2f(x)g(x)g'(0)$

$2g(x)g'(x) = 2g(x)f(x)g'(0).$

Adding, $2f(x)f'(x) + 2g(x)g'(x) = 0.$

Integrating, $f(x)^2 + g(x)^2 = C.$

Clearly $C \neq 0$, for if $C = 0$, then $f(x)^2 = -g(x)^2 \implies f(x) = g(x) = 0$, which contradicts that f, g are nonconstant.

$$\begin{aligned} \text{Now, } C = f(x+y)^2 + g(x+y)^2 &= (f(x)f(y) - g(x)g(y))^2 + (f(x)g(y) + g(x)f(y))^2 \\ &= f(x)^2f(y)^2 + g(x)^2g(y)^2 + f(x)^2g(y)^2 + g(x)^2f(y)^2 \\ &= [f(x)^2 + g(x)^2][f(y)^2 + g(y)^2] \\ &= C^2 \end{aligned}$$

Thus, $C = 1$ and we have $f(x)^2 + g(x)^2 = 1.$

Section 4.2 Area

1. $\displaystyle\sum_{i=1}^{5}(2i+1) = 2\sum_{i=1}^{5} i + \sum_{i=1}^{5} 1 = 2(1+2+3+4+5) + 5 = 35$

2. $\displaystyle\sum_{k=3}^{6} k(k-2) = 3(1) + 4(2) + 5(3) + 6(4) = 50$

3. $\displaystyle\sum_{k=0}^{4} \frac{1}{k^2+1} = 1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} = \frac{158}{85}$

4. $\displaystyle\sum_{j=3}^{5} \frac{1}{j} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$

5. $\displaystyle\sum_{k=1}^{4} c = c + c + c + c = 4c$

6. $\displaystyle\sum_{i=1}^{4}[(i-1)^2 + (i+1)^3] = (0+8) + (1+27) + (4+64) + (9+125) = 238$

7. $\displaystyle\sum_{i=1}^{9} \frac{1}{3i}$

8. $\displaystyle\sum_{i=1}^{15} \frac{5}{1+i}$

9. $\displaystyle\sum_{j=1}^{8}\left[5\left(\frac{j}{8}\right) + 3\right]$

10. $\displaystyle\sum_{j=1}^{4}\left[1 - \left(\frac{j}{4}\right)^2\right]$

11. $\displaystyle\frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^3 - \left(\frac{2i}{n}\right)\right]$

12. $\displaystyle\frac{2}{n}\sum_{i=1}^{n}\left[1 - \left(\frac{2i}{n} - 1\right)^2\right]$

13. $\displaystyle\frac{3}{n}\sum_{i=1}^{n}\left[2\left(1 + \frac{3i}{n}\right)^2\right]$

14. $\displaystyle\frac{1}{n}\sum_{i=0}^{n-1}\sqrt{1 - \left(\frac{i}{n}\right)^2}$

15. $\displaystyle\sum_{i=1}^{20} 2i = 2\sum_{i=1}^{20} i = 2\left[\frac{20(21)}{2}\right] = 420$

16. $\displaystyle\sum_{i=1}^{15}(2i-3) = 2\sum_{i=1}^{15} i - 3(15)$

$\displaystyle= 2\left[\frac{15(16)}{2}\right] - 45 = 195$

17. $\displaystyle\sum_{i=1}^{20}(i-1)^2 = \sum_{i=1}^{19} i^2$

$\displaystyle= \left[\frac{19(20)(39)}{6}\right] = 2470$

18. $\displaystyle\sum_{i=1}^{10}(i^2-1) = \sum_{i=1}^{10} i^2 - \sum_{i=1}^{10} 1$

$\displaystyle= \left[\frac{10(11)(21)}{6}\right] - 10 = 375$

19. $\sum_{i=1}^{15} i(i-1)^2 = \sum_{i=1}^{15} i^3 - 2\sum_{i=1}^{15} i^2 + \sum_{i=1}^{15} i$

$$= \frac{15^2(16)^2}{4} - 2\frac{15(16)(31)}{6} + \frac{15(16)}{2}$$

$$= 14{,}400 - 2480 + 120$$

$$= 12{,}040$$

20. $\sum_{i=1}^{10} i(i^2+1) = \sum_{i=1}^{10} i^3 + \sum_{i=1}^{10} i$

$$= \frac{10^2(11)^2}{4} + \left[\frac{10(11)}{2}\right] = 3080$$

21. sum seq(x ^ 2 + 3, x, 1, 20, 1) = 2930 (*TI-82*)

$$\sum_{i=1}^{20}(i^2+3) = \frac{20(20+1)(2(20)+1)}{6} + 3(20)$$

$$= \frac{(20)(21)(41)}{6} + 60 = 2930$$

22. sum seq(x ^ 3 − 2x, x, 1, 15, 1) = 14,160 (*TI-82*)

$$\sum_{i=1}^{15}(i^3-2i) = \frac{(15)^2(15+1)^2}{4} - 2\,\frac{15(15+1)}{2}$$

$$= \frac{(15)^2(16)^2}{4} - 15(16) = 14{,}160$$

23. $S = \left[3 + 4 + \frac{9}{2} + 5\right](1) = \frac{33}{2} = 16.5$

$s = \left[1 + 3 + 4 + \frac{9}{2}\right](1) = \frac{25}{2} = 12.5$

24. $S = [5 + 5 + 4 + 2](1) = 16$

$s = [4 + 4 + 2 + 0](1) = 10$

25. $S = [3 + 3 + 5](1) = 11$

$s = [2 + 2 + 3](1) = 7$

26. $S = \left[5 + 2 + 1 + \frac{2}{3} + \frac{1}{2}\right] = \frac{55}{6}$

$s = \left[2 + 1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{3}\right] = \frac{9}{2} = 4.5$

27. $S(4) = \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) + \sqrt{1}\left(\frac{1}{4}\right) = \frac{1+\sqrt{2}+\sqrt{3}+2}{8} \approx 0.768$

$s(4) = 0\left(\frac{1}{4}\right) + \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) = \frac{1+\sqrt{2}+\sqrt{3}}{8} \approx 0.518$

28. $S(8) = \left(\sqrt{\frac{1}{4}}+2\right)\frac{1}{4} + \left(\sqrt{\frac{1}{2}}+2\right)\frac{1}{4} + \left(\sqrt{\frac{3}{4}}+2\right)\frac{1}{4} + \left(\sqrt{1}+2\right)\frac{1}{4}$

$$+ \left(\sqrt{\frac{5}{4}}+2\right)\frac{1}{4} + \left(\sqrt{\frac{3}{2}}+2\right)\frac{1}{4} + \left(\sqrt{\frac{7}{4}}+2\right)\frac{1}{4} + \left(\sqrt{2}+2\right)\frac{1}{4}$$

$$= \frac{1}{4}\left(16 + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{5}}{2} + \frac{\sqrt{6}}{2} + \frac{\sqrt{7}}{2} + \sqrt{2}\right) \approx 6.038$$

$s(8) = (0+2)\frac{1}{4} + \left(\sqrt{\frac{1}{4}}+2\right)\frac{1}{4} + \left(\sqrt{\frac{1}{2}}+2\right)\frac{1}{4} + \cdots + \left(\sqrt{\frac{7}{4}}+2\right)\frac{1}{4} \approx 5.685$

29. $S(5) = 1\left(\frac{1}{5}\right) + \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \approx 0.746$

$s(5) = \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5}\right) = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \approx 0.646$

30. $S(5) = 1\left(\frac{1}{5}\right) + \sqrt{1-\left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1-\left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1-\left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1-\left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right)$

$$= \frac{1}{5}\left[1 + \frac{\sqrt{24}}{5} + \frac{\sqrt{21}}{5} + \frac{\sqrt{16}}{5} + \frac{\sqrt{9}}{5}\right] \approx 0.859$$

$s(5) = \sqrt{1-\left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1-\left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1-\left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1-\left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right) + 0 \approx 0.659$

31. $\lim_{n\to\infty}\left[\left(\frac{81}{n^4}\right)\frac{n^2(n+1)^2}{4}\right] = \frac{81}{4}\lim_{n\to\infty}\left[\frac{n^4+2n^3+n^2}{n^4}\right] = \frac{81}{4}(1) = \frac{81}{4}$

32. $\lim_{n\to\infty}\left[\left(\frac{64}{n^3}\right)\frac{n(n+1)(2n+1)}{6}\right] = \frac{64}{6}\lim_{n\to\infty}\left[\frac{2n^3+3n^2+n}{n^3}\right] = \frac{64}{6}(2) = \frac{64}{3}$

33. $\lim_{n\to\infty}\left[\left(\frac{18}{n^2}\right)\frac{n(n+1)}{2}\right] = \frac{18}{2}\lim_{n\to\infty}\left[\frac{n^2+n}{n^2}\right] = \frac{18}{2}(1) = 9$

34. $\lim_{n\to\infty}\left[\left(\frac{1}{n^2}\right)\frac{n(n+1)}{2}\right] = \frac{1}{2}\lim_{n\to\infty}\left[\frac{n^2+n}{n^2}\right] = \frac{1}{2}(1) = \frac{1}{2}$

35. $\sum_{i=1}^{n}\frac{2i+1}{n^2} = \frac{1}{n^2}\sum_{i=1}^{n}(2i+1) = \frac{1}{n^2}\left[2\frac{n(n+1)}{2}+n\right] = \frac{n+2}{n} = 1+\frac{2}{n} = S(n)$

$S(10) = \frac{12}{10} = 1.2$

$S(100) = 1.02$

$S(1000) = 1.002$

$S(10{,}000) = 1.0002$

36. $\sum_{j=1}^{n}\frac{4j+3}{n^2} = \frac{1}{n^2}\sum_{j=1}^{n}(4j+3) = \frac{1}{n^2}\left[\frac{4n(n+1)}{2}+3n\right] = \frac{2n+5}{n} = S(n)$

$S(10) = \frac{25}{10} = 2.5$

$S(100) = 2.05$

$S(1000) = 2.005$

$S(10{,}000) = 2.0005$

37. $\sum_{k=1}^{n}\frac{6k(k-1)}{n^3} = \frac{6}{n^3}\sum_{k=1}^{n}(k^2-k) = \frac{6}{n^3}\left[\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}\right]$

$= \frac{6}{n^2}\left[\frac{2n^2+3n+1-3n-3}{6}\right] = \frac{1}{n^2}[2n^2-2] = 2-\frac{2}{n^2} = S(n)$

$S(10) = 1.98$

$S(100) = 1.9998$

$S(1000) = 1.999998$

$S(10{,}000) = 1.99999998$

38. $\sum_{i=1}^{n}\frac{4i^2(i-1)}{n^4} = \frac{4}{n^4}\sum_{i=1}^{n}(i^3-i^2) = \frac{4}{n^4}\left[\frac{n^2(n+1)^2}{4}-\frac{n(n+1)(2n+1)}{6}\right]$

$= \frac{4}{n^3}\left[\frac{n^3+2n^2+n}{4}-\frac{2n^2+3n+1}{6}\right]$

$= \frac{1}{3n^3}[3n^3+6n^2+3n-4n^2-6n-2]$

$= \frac{1}{3n^3}[3n^3+2n^2-3n-2] = S(n)$

$S(10) = 1.056$

$S(100) = 1.006566$

$S(1000) = 1.00066567$

$S(10{,}000) = 1.000066657$

39. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{16i}{n^2}\right) = \lim_{n\to\infty}\frac{16}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty}\frac{16}{n^2}\left(\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\left[8\left(\frac{n^2+n}{n^2}\right)\right] = 8\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 8$

40. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty}\frac{4}{n^2}\left(\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\frac{4}{2}\left(1+\frac{1}{n}\right) = 2$

41. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n^3}(i-1)^2 = \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n-1} i^2 = \lim_{n\to\infty}\frac{1}{n^3}\left[\frac{(n-1)(n)(2n-1)}{6}\right]$

$= \lim_{n\to\infty}\frac{1}{6}\left[\frac{2n^3-3n^2+n}{n^3}\right] = \lim_{n\to\infty}\left[\frac{1}{6}\left(\frac{2-(3/n)+(1/n^2)}{1}\right)\right] = \frac{1}{3}$

42. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{2i}{n}\right)^2\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{2}{n^3}\sum_{i=1}^{n}(n+2i)^2$

$= \lim_{n\to\infty}\frac{2}{n^3}\left[\sum_{i=1}^{n} n^2 + 4n\sum_{i=1}^{n} i + 4\sum_{i=1}^{n} i^2\right]$

$= \lim_{n\to\infty}\frac{2}{n^3}\left[n^3 + (4n)\left(\frac{n(n+1)}{2}\right) + \frac{4(n)(n+1)(2n+1)}{6}\right]$

$= 2\lim_{n\to\infty}\left[1+2+\frac{2}{n}+\frac{4}{3}+\frac{2}{n}+\frac{2}{3n^2}\right]$

$= 2\left(1+2+\frac{4}{3}\right) = \frac{26}{3}$

43. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)\left(\frac{2}{n}\right) = 2\lim_{n\to\infty}\frac{1}{n}\left[\sum_{i=1}^{n} 1 + \frac{1}{n}\sum_{i=1}^{n} i\right] = 2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{1}{n}\left(\frac{n(n+1)}{2}\right)\right] = 2\lim_{n\to\infty}\left[1+\frac{n^2+n}{2n^2}\right] = 2\left(1+\frac{1}{2}\right) = 3$

44. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{2i}{n}\right)^3\left(\frac{2}{n}\right) = 2\lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}(n+2i)^3$

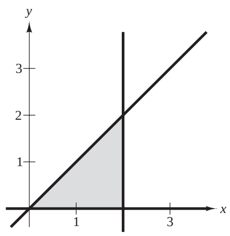
$= 2\lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}(n^3+6n^2i+12ni^2+8i^3)$

$= 2\lim_{n\to\infty}\frac{1}{n^4}\left[n^4+6n^2\left(\frac{n(n+1)}{2}\right)+12n\left(\frac{n(n+1)(2n+1)}{6}\right)+8\left(\frac{n^2(n+1)^2}{4}\right)\right]$

$= 2\lim_{n\to\infty}\left(1+3+\frac{3}{n}+4+\frac{6}{n}+\frac{2}{n^2}+2+\frac{4}{n}+\frac{2}{n^2}\right)$

$= 2\lim_{n\to\infty}\left(10+\frac{13}{n}+\frac{4}{n^2}\right) = 20$

45. (a)



(b) $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

Endpoints:

$0 < 1\left(\frac{2}{n}\right) < 2\left(\frac{2}{n}\right) < \cdots < (n-1)\left(\frac{2}{n}\right) < n\left(\frac{2}{n}\right) = 2$

—CONTINUED—

45. —CONTINUED—

(c) Since $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$s(n) = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$= \sum_{i=1}^{n} f\left(\frac{2i-2}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\frac{2}{n} = \sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(e)

x	5	10	50	100
$s(n)$	1.6	1.8	1.96	1.98
$S(n)$	2.4	2.2	2.04	2.02

(f) $$\lim_{n\to\infty}\sum_{i=1}^{n}\left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n}(i-1)$$

$$= \lim_{n\to\infty}\frac{4}{n^2}\left[\frac{n(n+1)}{2} - n\right]$$

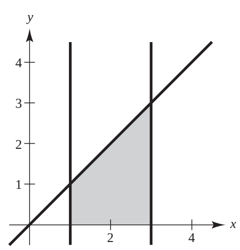
$$= \lim_{n\to\infty}\left[\frac{2(n+1)}{n} - \frac{4}{n}\right] = 2$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty}\left(\frac{4}{n^2}\right)\frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty}\frac{2(n+1)}{n} = 2$$

46. (a)



(b) $\Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}$

Endpoints:

$$1 < 1 + \frac{2}{n} < 1 + \frac{4}{n} < \cdots < 1 + \frac{2n}{n} = 3$$

$$1 < 1 + 1\left(\frac{2}{n}\right) < 1 + 2\left(\frac{2}{n}\right) < \cdots < 1 + (n-1)\left(\frac{2}{n}\right) < 1 + n\left(\frac{2}{n}\right)$$

(c) Since $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$s(n) = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$= \sum_{i=1}^{n} f\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(e)

x	5	10	50	100
$s(n)$	3.6	3.8	3.96	3.98
$S(n)$	4.4	4.2	4.04	4.02

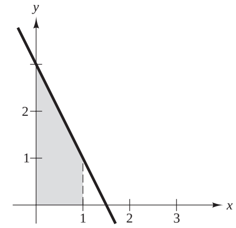
(f) $$\lim_{n\to\infty}\sum_{i=1}^{n}\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\left(\frac{2}{n}\right)\left[n + \frac{2}{n}\left(\frac{n(n+1)}{2} - n\right)\right]$$

$$= \lim_{n\to\infty}\left[2 + \frac{2n+2}{n} - \frac{4}{n}\right] = \lim_{n\to\infty}\left[4 - \frac{2}{n}\right] = 4$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{2}{n}\left[n + \left(\frac{2}{n}\right)\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[2 + \frac{2(n+1)}{n}\right] = \lim_{n\to\infty}\left[4 + \frac{2}{n}\right] = 4$$

47. $y = -2x + 3$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}\right)$

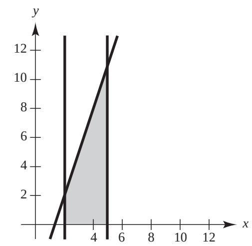


$$s(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[-2\left(\frac{i}{n}\right) + 3\right]\left(\frac{1}{n}\right)$$

$$= 3 - \frac{2}{n^2}\sum_{i=1}^{n} i = 3 - \frac{2(n+1)n}{2n^2} = 2 - \frac{1}{n}$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 2$$

48. $y = 3x - 4$ on $[2, 5]$. $\left(\textbf{Note: } \Delta x = \dfrac{5-2}{n} = \dfrac{3}{n}\right)$



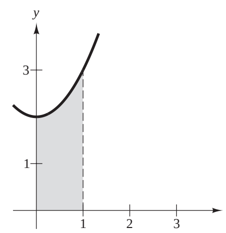
$$S(n) = \sum_{i=1}^{n} f\left(2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \sum_{i=1}^{n}\left[3\left(2 + \frac{3i}{n}\right) - 4\right]\left(\frac{3}{n}\right) = 18 + 3\left(\frac{3}{n}\right)^2\sum_{i=1}^{n} i - 12$$

$$= 6 + \frac{27}{n^2}\left(\frac{(n+1)n}{2}\right) = 6 + \frac{27}{2}\left(1 + \frac{1}{n}\right)$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 6 + \frac{27}{2} = \frac{39}{2}$$

49. $y = x^2 + 2$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1}{n}\right)$

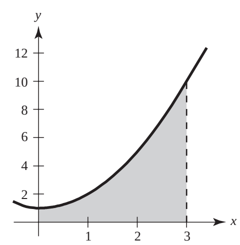


$$S(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^2 + 2\right]\left(\frac{1}{n}\right)$$

$$= \left[\frac{1}{n^3}\sum_{i=1}^{n} i^2\right] + 2 = \frac{n(n+1)(2n+1)}{6n^3} + 2 = \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 2$$

$$\text{Area} = \lim_{n\to\infty} S(n) = \frac{7}{3}$$

50. $y = x^2 + 1$ on $[0, 3]$. $\left(\textbf{Note: } \Delta x = \dfrac{3-0}{n} = \dfrac{3}{n}\right)$



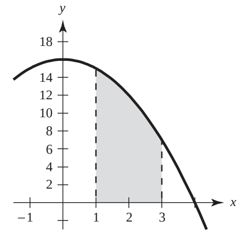
$$S(n) = \sum_{l=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{l=1}^{n}\left[\left(\frac{3i}{n}\right)^2 + 1\right]\left(\frac{3}{n}\right)$$

$$= \frac{27}{n^3}\sum_{l=1}^{n} l^2 + \frac{3}{n}\sum_{l=1}^{n} 1$$

$$= \frac{27}{n^3}\,\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}(n) = \frac{9}{2}\,\frac{2n^2 + 3n + 1}{n^2} + 3$$

$$\text{Area} = \lim_{n\to\infty} S(n) = \frac{9}{2}(2) + 3 = 12$$

51. $y = 16 - x^2$ on $[1, 3]$. $\left(\textbf{Note: } \Delta x = \frac{2}{n}\right)$



$$s(n) = \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[16 - \left(1 + \frac{2i}{n}\right)^2\right]\left(\frac{2}{n}\right)$$

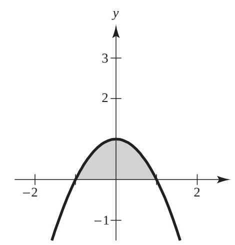
$$= \frac{2}{n}\sum_{i=1}^{n}\left[15 - \frac{4i^2}{n^2} - \frac{4i}{n}\right]$$

$$= \frac{2}{n}\left[15n - \frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{4}{n}\frac{n(n+1)}{2}\right]$$

$$= 30 - \frac{8}{6n^2}(n+1)(2n+1) - \frac{4}{n}(n+1)$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 30 - \frac{8}{3} - 4 = \frac{70}{3} = 23\frac{1}{3}$$

52. $y = 1 - x^2$ on $[-1, 1]$. Find area of region over the interval $[0, 1]$. $\left(\textbf{Note: } \Delta x = \frac{1}{n}\right)$



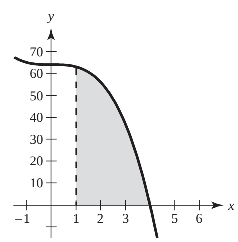
$$s(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[1 - \left(\frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right)$$

$$= 1 - \frac{1}{n^3}\sum_{i=1}^{n} i^2 = 1 - \frac{n(n+1)(2n+1)}{6n^3} = 1 - \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$\frac{1}{2}\text{Area} = \lim_{n\to\infty} s(n) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\text{Area} = \frac{4}{3}$$

53. $y = 64 - x^3$ on $[1, 4]$. $\left(\textbf{Note: } \Delta x = \frac{4-1}{n} = \frac{3}{n}\right)$



$$s(n) = \sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^{n}\left[64 - \left(1 + \frac{3i}{n}\right)^3\right]\left(\frac{3}{n}\right)$$

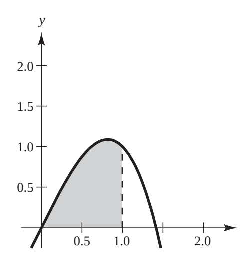
$$= \frac{3}{n}\sum_{i=1}^{n}\left[63 - \frac{27i^3}{n^3} - \frac{27i^2}{n^2} - \frac{9i}{n}\right]$$

$$= \frac{3}{n}\left[63n - \frac{27}{n^3}\frac{n^2(n+1)^2}{4} - \frac{27}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{9}{n}\frac{n(n+1)}{2}\right]$$

$$= 189 - \frac{81}{4n^2}(n+1)^2 - \frac{81}{6n^2}(n+1)(2n+1) - \frac{27}{2}\frac{n+1}{n}$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 189 - \frac{81}{4} - 27 - \frac{27}{2} = \frac{513}{4} = 128.25$$

54. $y = 2x - x^3$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \frac{1-0}{n} = \frac{1}{n}\right)$



Since y both increases and decreases on $[0, 1]$, $T(n)$ is neither an upper nor lower sum.

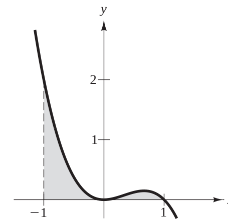
$$T(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[2\left(\frac{i}{n}\right) - \left(\frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right)$$

$$= \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^4}\sum_{i=1}^{n} i^3 = \frac{n(n+1)}{n^2} - \frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right]$$

$$= 1 + \frac{1}{n} - \frac{1}{4} - \frac{2}{4n} - \frac{1}{4n^2}$$

$$\text{Area} = \lim_{n\to\infty} T(n) = 1 - \frac{1}{4} = \frac{3}{4}$$

55. $y = x^2 - x^3$ on $[-1, 1]$. $\left(\textbf{Note: } \Delta x = \frac{1-(-1)}{n} = \frac{2}{n}\right)$



Again, $T(n)$ is neither an upper nor a lower sum.

$$T(n) = \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[\left(-1 + \frac{2i}{n}\right)^2 - \left(-1 + \frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left[\left(1 - \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right)\right]\left(\frac{2}{n}\right)$$

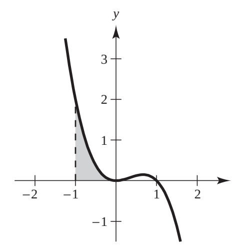
$$= \sum_{i=1}^{n}\left[2 - \frac{10i}{n} + \frac{16i^2}{n^2} - \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right) = \frac{4}{n}\sum_{i=1}^{n} 1 - \frac{20}{n^2}\sum_{i=1}^{n} i + \frac{32}{n^3}\sum_{i=1}^{n} i^2 - \frac{16}{n^4}\sum_{i=1}^{n} i^3$$

$$= \frac{4}{n}(n) - \frac{20}{n^2} \cdot \frac{n(n+1)}{2} + \frac{32}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{16}{n^4} \cdot \frac{n^2(n+1)^2}{4}$$

$$= 4 - 10\left(1 + \frac{1}{n}\right) + \frac{16}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) - 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$$

$$\text{Area} = \lim_{n\to\infty} T(n) = 4 - 10 + \frac{32}{3} - 4 = \frac{2}{3}$$

56. $y = x^2 - x^3$ on $[-1, 0]$. $\left(\textbf{Note: } \Delta x = \frac{0-(-1)}{n} = \frac{1}{n}\right)$



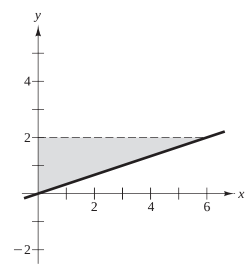
$$s(n) = \sum_{i=1}^{n} f\left(-1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[\left(-1 + \frac{i}{n}\right)^2 - \left(-1 + \frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n}\left[\left(2 - \frac{5i}{n} + \frac{4i^2}{n^2} - \frac{i^3}{n^3}\right)\left(\frac{1}{n}\right)\right] = 2 - \frac{5}{n^2}\sum_{i=1}^{n} i + \frac{4}{n^3}\sum_{i=1}^{n} i^2 - \frac{1}{n^4}\sum_{i=1}^{n} i^3$$

$$= 2 - \frac{5}{2} - \frac{5}{2n} + \frac{4}{3} + \frac{2}{n} + \frac{2}{3n^3} - \frac{1}{4} - \frac{1}{2n} - \frac{1}{4n^2}$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 2 - \frac{5}{2} + \frac{4}{3} - \frac{1}{4} = \frac{7}{12}$$

57. $f(y) = 3y,\ 0 \le y \le 2$ $\left(\textbf{Note: } \Delta y = \frac{2-0}{n} = \frac{2}{n}\right)$

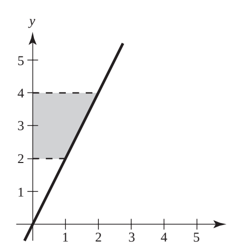


$$S(n) = \sum_{i=1}^{n} f(m_i)\,\Delta y = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n} 3\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \frac{12}{n^2}\sum_{i=1}^{n} i = \left(\frac{12}{n^2}\right) \cdot \frac{n(n+1)}{2} = \frac{6(n+1)}{n} = 6 + \frac{6}{n}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(6 + \frac{6}{n}\right) = 6$$

58. $g(y) = \frac{1}{2}y,\ 2 \le y \le 4.$ $\left(\textbf{Note: } \Delta y = \frac{4-2}{n} = \frac{2}{n}\right)$



$$S(n) = \sum_{i=1}^{n} g\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n} \frac{1}{2}\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \frac{2}{n}\sum_{i=1}^{n}\left(1 + \frac{i}{n}\right)$$

$$= \frac{2}{n}\left[n + \frac{1}{n}\frac{n(n+1)}{2}\right] = 2 + \frac{n+1}{n}$$

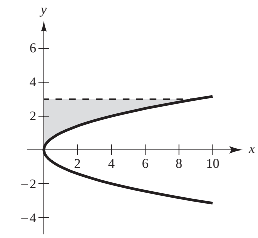
$$\text{Area} = \lim_{n\to\infty} S(n) = 2 + 1 = 3$$

59. $f(y) = y^2, 0 \le y \le 3 \quad \left(\textbf{Note: } \Delta y = \frac{3-0}{n} = \frac{3}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^{n}\left(\frac{3i}{n}\right)^2\left(\frac{3}{n}\right) = \frac{27}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{9}{n^2}\left(\frac{2n^2+3n+1}{2}\right) = 9 + \frac{27}{2n} + \frac{9}{2n^2}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(9 + \frac{27}{2n} + \frac{9}{2n^2}\right) = 9$$



60. $f(y) = 4y - y^2, 1 \le y \le 2. \left(\textbf{Note: } \Delta y = \frac{2-1}{n} = \frac{1}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[4\left(1+\frac{i}{n}\right) - \left(1+\frac{i}{n}\right)^2\right]$$

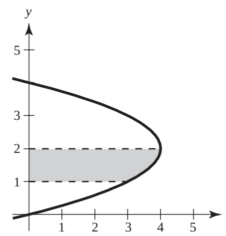
$$= \frac{1}{n}\sum_{i=1}^{n}\left(4 + \frac{4i}{n} - 1 - \frac{2i}{n} - \frac{i^2}{n^2}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(3 + \frac{2i}{n} - \frac{i^2}{n^2}\right)$$

$$= \frac{1}{n}\left[3n + \frac{2}{n}\frac{n(n+1)}{2} - \frac{1}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 3 + \frac{n+1}{n} - \frac{(n+1)(2n+1)}{6}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 3 + 1 - \frac{1}{3} = \frac{11}{3}$$



61. $g(y) = 4y^2 - y^3, 1 \le y \le 3. \quad \left(\textbf{Note: } \Delta y = \frac{3-1}{n} = \frac{2}{n}\right)$

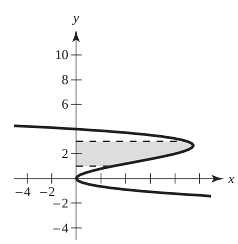
$$S(n) = \sum_{i=1}^{n} g\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left[4\left(1+\frac{2i}{n}\right)^2 - \left(1+\frac{2i}{n}\right)^3\right]\frac{2}{n}$$

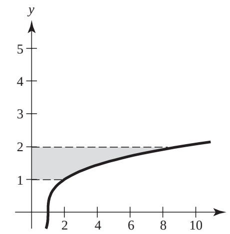
$$= \frac{2}{n}\sum_{i=1}^{n} 4\left[1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right] - \left[1 + \frac{6i}{n} + \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right]$$

$$= \frac{2}{n}\sum_{i=1}^{n}\left[3 + \frac{10i}{n} + \frac{4i^2}{n^2} - \frac{8i^3}{n^3}\right] = \frac{2}{n}\left[3n + \frac{10}{n}\frac{n(n+1)}{2} + \frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{8}{n^2}\frac{n^2(n+1)^2}{4}\right]$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 6 + 10 + \frac{8}{3} - 4 = \frac{44}{3}$$



62. $h(y) = y^3 + 1,\ 1 \le y \le 2 \left(\textbf{Note: } \Delta y = \frac{1}{n}\right)$



$$S(n) = \sum_{i=1}^{n} h\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right)$$
$$= \sum_{i=1}^{n}\left[\left(1 + \frac{i}{n}\right)^3 + 1\right]\frac{1}{n}$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left(2 + \frac{i^3}{n^3} + \frac{3i^2}{n^2} + \frac{3i}{n}\right)$$
$$= \frac{1}{n}\left[2n + \frac{1}{n^3}\frac{n^2(n+1)^2}{4} + \frac{3}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}\frac{n(n+1)}{2}\right]$$
$$= 2 + \frac{(n+1)^2}{n^2 4} + \frac{1}{2}\frac{(n+1)(2n+1)}{n^2} + \frac{3(n+1)}{2n}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 2 + \frac{1}{4} + 1 + \frac{3}{2} = \frac{19}{4}$$

63. $f(x) = x^2 + 3,\ 0 \le x \le 2,\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$$\Delta x = \frac{1}{2},\ c_1 = \frac{1}{4},\ c_2 = \frac{3}{4},\ c_3 = \frac{5}{4},\ c_4 = \frac{7}{4}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4}\left[c_i^2 + 3\right]\left(\frac{1}{2}\right)$$
$$= \frac{1}{2}\left[\left(\frac{1}{16} + 3\right) + \left(\frac{9}{16} + 3\right) + \left(\frac{25}{16} + 3\right) + \left(\frac{49}{16} + 3\right)\right]$$
$$= \frac{69}{8}$$

64. $f(x) = x^2 + 4x,\ 0 \le x \le 4,\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$$\Delta x = 1,\ c_1 = \frac{1}{2},\ c_2 = \frac{3}{2},\ c_3 = \frac{5}{2},\ c_4 = \frac{7}{2}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4}\left[c_i^2 + 4c_i\right](1)$$
$$= \left[\left(\frac{1}{4} + 2\right) + \left(\frac{9}{4} + 6\right) + \left(\frac{25}{4} + 10\right) + \left(\frac{49}{4} + 14\right)\right]$$
$$= 53$$

65. $f(x) = \tan x,\ 0 \le x \le \frac{\pi}{4},\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$$\Delta x = \frac{\pi}{16},\ c_1 = \frac{\pi}{32},\ c_2 = \frac{3\pi}{32},\ c_3 = \frac{5\pi}{32},\ c_4 = \frac{7\pi}{32}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4}(\tan c_i)\left(\frac{\pi}{16}\right)$$
$$= \frac{\pi}{16}\left(\tan\frac{\pi}{32} + \tan\frac{3\pi}{32} + \tan\frac{5\pi}{32} + \tan\frac{7\pi}{32}\right) \approx 0.345$$

66. $f(x) = \sin x,\ 0 \le x \le \frac{\pi}{2},\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$$\Delta x = \frac{\pi}{8},\ c_1 = \frac{\pi}{16},\ c_2 = \frac{3\pi}{16},\ c_3 = \frac{5\pi}{16},\ c_4 = \frac{7\pi}{16}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4}(\sin c_i)\left(\frac{\pi}{8}\right)$$
$$= \frac{\pi}{8}\left(\sin\frac{\pi}{16} + \sin\frac{3\pi}{16} + \sin\frac{5\pi}{16} + \sin\frac{7\pi}{16}\right) \approx 1.006$$

67. $f(x) = \sqrt{x}$ on $[0, 4]$.

n	4	8	12	16	20
Approximate area	5.3838	5.3523	5.3439	5.3403	5.3384

(Exact value is 16/3.)

68. $f(x) = \dfrac{8}{x^2 + 1}$ on $[2, 6]$.

n	4	8	12	16	20
Approximate area	2.3397	2.3755	2.3824	2.3848	2.3860

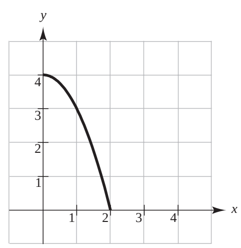
69. $f(x) = \tan\left(\dfrac{\pi x}{8}\right)$ on $[1, 3]$.

n	4	8	12	16	20
Approximate area	2.2223	2.2387	2.2418	2.2430	2.2435

70. $f(x) = \cos\sqrt{x}$ on $[0, 2]$.

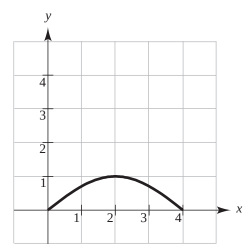
n	4	8	12	16	20
Approximate area	1.1041	1.1053	1.1055	1.1056	1.1056

71.



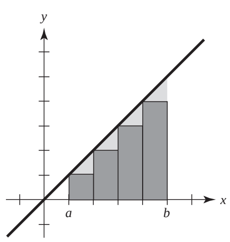
(b) $A \approx 6$ square units

72.

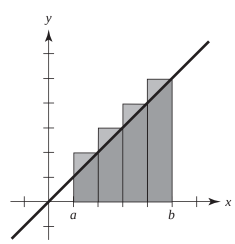


(a) $A \approx 3$ square units

73. We can use the line $y = x$ bounded by $x = a$ and $x = b$. The sum of the areas of these inscribed rectangles is the lower sum.



The sum of the areas of these circumscribed rectangles is the upper sum.

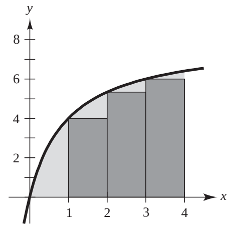


We can see that the rectangles do not contain all of the area in the first graph and the rectangles in the second graph cover more than the area of the region.

The exact value of the area lies between these two sums.

74. See the definition of area. Page 265.

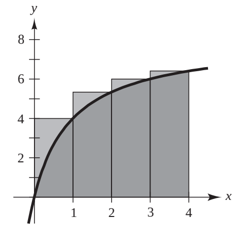
75. (a)



Lower sum:

$s(4) = 0 + 4 + 5\frac{1}{3} + 6 = 15\frac{1}{3} = \frac{46}{3} \approx 15.333$

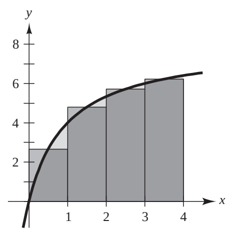
(b)



Upper sum:

$S(4) = 4 + 5\frac{1}{3} + 6 + 6\frac{2}{5} = 21\frac{11}{15} = \frac{326}{15} \approx 21.733$

(c)



Midpoint Rule:

$M(4) = 2\frac{2}{3} + 4\frac{4}{5} + 5\frac{5}{7} + 6\frac{2}{9} = \frac{6112}{315} \approx 19.403$

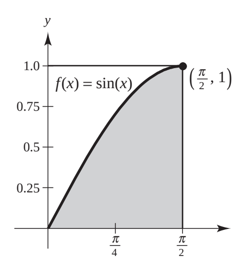
(d) In each case, $\Delta x = 4/n$. The lower sum uses left endpoints, $(i - 1)(4/n)$. The upper sum uses right endpoints, $(i)(4/n)$. The Midpoint Rule uses midpoints, $\left(i - \frac{1}{2}\right)(4/n)$.

(e)

n	4	8	20	100	200
$s(n)$	15.333	17.368	18.459	18.995	19.06
$S(n)$	21.733	20.568	19.739	19.251	19.188
$M(n)$	19.403	19.201	19.137	19.125	19.125

(f) $s(n)$ increases because the lower sum approaches the exact value as n increases. $S(n)$ decreases because the upper sum approaches the exact value as n increases. Because of the shape of the graph, the lower sum is always smaller than the exact value, whereas the upper sum is always larger.

76. $f(x) = \sin x, \left[0, \frac{\pi}{2}\right]$



Let A_1 = area bounded by $f(x) = \sin x$, the x-axis, $x = 0$ and $x = \pi/2$. Let A_2 = area of the rectangle bounded by $y = 1$, $y = 0$, $x = 0$, and $x = \pi/2$. Thus, $A_2 = (\pi/2)(1) \approx 1.570796$. In this program, the computer is generating N_2 pairs of random points in the rectangle whose area is represented by A_2. It is keeping track of how many of these points, N_1, lie in the region whose area is represented by A_1. Since the points are randomly generated, we assume that

$$\frac{A_1}{A_2} \approx \frac{N_1}{N_2} \Longrightarrow A_1 \approx \frac{N_1}{N_2}A_2.$$

The larger N_2 is the better the approximation to A_1.

77. True. (Theorem 4.2 (2))

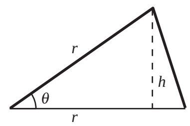
78. True. (Theorem 4.3)

79. Suppose there are n rows and $n + 1$ columns in the figure. The stars on the left total $1 + 2 + \cdots + n$, as do the stars on the right. There are $n(n + 1)$ stars in total, hence

$$2[1 + 2 + \cdots + n] = n(n + 1)$$

$$1 + 2 + \cdots + n = \frac{1}{2}(n)(n + 1).$$

80. (a) $\theta = \dfrac{2\pi}{n}$



(b) $\sin\theta = \dfrac{h}{r}$

$$h = r\sin\theta$$

$$A = \frac{1}{2}bh = \frac{1}{2}r(r\sin\theta) = \frac{1}{2}r^2\sin\theta$$

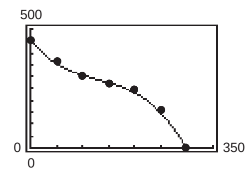
(c) $A_n = n\left(\dfrac{1}{2}r^2\sin\dfrac{2\pi}{n}\right) = \dfrac{r^2n}{2}\sin\dfrac{2\pi}{n} = \pi r^2\left(\dfrac{\sin(2\pi/n)}{2\pi/n}\right)$

Let $x = 2\pi/n$. As $n \to \infty$, $x \to 0$.

$$\lim_{n\to\infty} A_n = \lim_{x\to 0} \pi r^2\left(\frac{\sin x}{x}\right) = \pi r^2(1) = \pi r^2$$

81. (a) $y = (-4.09 \times 10^{-5})x^3 + 0.016x^2 - 2.67x + 452.9$

(b)



(c) Using the integration capability of a graphing utility, you obtain

$$A \approx 76{,}897.5 \text{ ft}^2.$$

82. For n odd,

$n = 1,$	1 row,	1 block
$n = 3,$	2 rows,	4 blocks
$n = 5,$	3 rows,	9 blocks
$n,$	$\dfrac{n+1}{2}$ rows,	$\left(\dfrac{n+1}{2}\right)^2$ blocks,

For n even,

$n = 2,$	1 row,	2 block
$n = 4,$	2 rows,	6 blocks
$n = 6,$	3 rows,	12 blocks
$n,$	$\dfrac{n}{2}$ rows,	$\dfrac{n^2+2n}{4}$ blocks,

83. (a) $\displaystyle\sum_{i=1}^{n} 2i = n(n+1)$

The formula is true for $n = 1$: $2 = 1(1+1) = 2$.

Assume that the formula is true for $n = k$:

$$\sum_{i=1}^{k} 2i = k(k+1).$$

Then we have
$$\sum_{i=1}^{k+1} 2i = \sum_{i=1}^{k} 2i + 2(k+1)$$
$$= k(k+1) + 2(k+1)$$
$$= (k+1)(k+2)$$

which shows that the formula is true for $n = k+1$.

(b) $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

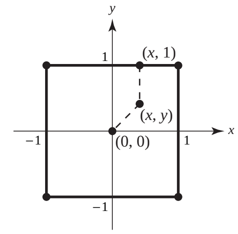
The formula is true for $n = 1$ because
$$1^3 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1.$$

Assume that the formula is true for $n = k$:
$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}.$$

Then we have
$$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$$
$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$
$$= \frac{(k+1)^2}{4}[k^2 + 4(k+1)]$$
$$= \frac{(k+1)^2}{4}(k+2)^2$$

which shows that the formula is true for $n = k+1$.

84. Assume that the dartboard has corners at $(\pm 1, \pm 1)$.



A point (x, y) in the square is closer to the center than the top edge if

$$\sqrt{x^2 + y^2} \le 1 - y$$

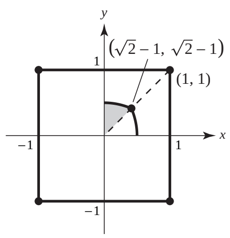
$$x^2 + y^2 \le 1 - 2y + y^2$$

$$y \le \frac{1}{2}(1 - x^2).$$

By symmetry, a point (x, y) in the square is closer to the center than the right edge if

$$x \le \frac{1}{2}(1 - y^2).$$

In the first quadrant, the parabolas $y = \frac{1}{2}(1 - x^2)$ and $x = \frac{1}{2}(1 - y^2)$ intersect at $\left(\sqrt{2} - 1, \sqrt{2} - 1\right)$. There are 8 equal regions that make up the total region, as indicated in the figure.

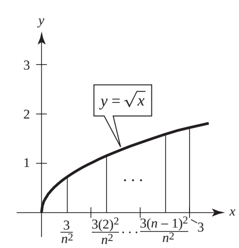


$$\text{Area of shaded region } S = \int_0^{\sqrt{2}-1} \left[\frac{1}{2}(1 - x^2) - x\right] dx = \frac{2\sqrt{2}}{3} - \frac{5}{6}$$

$$\text{Probability} = \frac{8S}{\text{Area square}} = 2\left[\frac{2\sqrt{2}}{3} - \frac{5}{6}\right] = \frac{4\sqrt{2}}{3} - \frac{5}{3}$$

Section 4.3 Riemann Sums and Definite Integrals

1. $f(x) = \sqrt{x}, y = 0, x = 0, x = 3, c_i = \dfrac{3i^2}{n^2}$



$$\Delta x_i = \frac{3i^2}{n^2} - \frac{3(i - 1)^2}{n^2} = \frac{3}{n^2}(2i - 1)$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{\frac{3i^2}{n^2}}\,\frac{3}{n^2}(2i - 1)$$

$$= \lim_{n \to \infty} \frac{3\sqrt{3}}{n^3} \sum_{i=1}^{n} (2i^2 - i)$$

$$= \lim_{n \to \infty} \frac{3\sqrt{3}}{n^3}\left[2\frac{n(n + 1)(2n + 1)}{6} - \frac{n(n + 1)}{2}\right]$$

$$= \lim_{n \to \infty} 3\sqrt{3}\left[\frac{(n + 1)(2n + 1)}{3n^2} - \frac{n + 1}{2n^2}\right]$$

$$= 3\sqrt{3}\left[\frac{2}{3} - 0\right] = 2\sqrt{3} \approx 3.464$$

2. $f(x) = \sqrt[3]{x}, y = 0, x = 0, x = 1, c_i = \dfrac{i^3}{n^3}$

$$\Delta x_i = \frac{i^3}{n^3} - \frac{(i-1)^3}{n^3} = \frac{3i^2 - 3i + 1}{n^3}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt[3]{\frac{i^3}{n^3}}\left[\frac{3i^2 - 3i + 1}{n^3}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} (3i^3 - 3i^2 + i)$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\left[3\left(\frac{n^2(n+1)^2}{4}\right) - 3\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\left[\frac{3n^4 + 6n^3 + 3n^2}{4} - \frac{2n^3 + 3n^2 + n}{2} + \frac{n^2 + n}{2}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\left[\frac{3n^4}{4} + \frac{n^3}{2} - \frac{n^2}{4}\right] = \lim_{n\to\infty}\left[\frac{3}{4} + \frac{1}{2n} - \frac{1}{4n^2}\right] = \frac{3}{4}$$

3. $y = 6$ on $[4, 10]$. $\left(\textbf{Note}\text{: } \Delta x = \dfrac{10-4}{n} = \dfrac{6}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(4 + \frac{6i}{n}\right)\left(\frac{6}{n}\right) = \sum_{i=1}^{n} 6\left(\frac{6}{n}\right) = \sum_{i=1}^{n} \frac{36}{n} = \frac{1}{n}\sum_{i=1}^{n} 36 = \frac{1}{n}(36n) = 36$$

$$\int_4^{10} 6\,dx = \lim_{n\to\infty} 36 = 36$$

4. $y = x$ on $[-2, 3]$. $\left(\textbf{Note:}\ \Delta x = \dfrac{3-(-2)}{n} = \dfrac{5}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(-2 + \frac{5i}{n}\right)\left(\frac{5}{n}\right) = \sum_{i=1}^{n}\left(-2 + \frac{5i}{n}\right)\left(\frac{5}{n}\right) = -10 + \frac{25}{n^2}\sum_{i=1}^{n} i$$

$$= -10 + \left(\frac{25}{n^2}\right)\frac{n(n+1)}{2} = -10 + \frac{25}{2}\left(1 + \frac{1}{n}\right) = \frac{5}{2} + \frac{25}{2n}$$

$$\int_{-2}^{3} x\,dx = \lim_{n\to\infty}\left(\frac{5}{2} + \frac{25}{2n}\right) = \frac{5}{2}$$

5. $y = x^3$ on $[-1, 1]$. $\left(\textbf{Note:}\ \Delta x = \dfrac{1-(-1)}{n} = \dfrac{2}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left(-1 + \frac{2i}{n}\right)^3\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right)$$

$$= -2 + \frac{12}{n^2}\sum_{i=1}^{n} i - \frac{24}{n^3}\sum_{i=1}^{n} i^2 + \frac{16}{n^4}\sum_{i=1}^{n} i^3$$

$$= -2 + 6\left(1 + \frac{1}{n}\right) - 4\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{2}{n}$$

$$\int_{-1}^{1} x^3\,dx = \lim_{n\to\infty} \frac{2}{n} = 0$$

6. $y = 3x^2$ on $[1, 3]$. $\left(\text{Note: } \Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n} 3\left(1 + \frac{2i}{n}\right)^2\left(\frac{2}{n}\right)$$

$$= \frac{6}{n}\sum_{i=1}^{n}\left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right)$$

$$= \frac{6}{n}\left[n + \frac{4}{n}\frac{n(n+1)}{2} + \frac{4}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 6 + 12\frac{n+1}{n} + 4\frac{(n+1)(2n+1)}{n^2}$$

$$\int_1^3 3x^2\,dx = \lim_{n\to\infty}\left[6 + \frac{12(n+1)}{n} + \frac{4(n+1)(2n+1)}{n^2}\right]$$

$$= 6 + 12 + 8 = 26$$

7. $y = x^2 + 1$ on $[1, 2]$. $\left(\textbf{Note: } \Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[\left(1 + \frac{i}{n}\right)^2 + 1\right]\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[1 + \frac{2i}{n} + \frac{i^2}{n^2} + 1\right]\left(\frac{1}{n}\right)$$

$$= 2 + \frac{2}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2 = 2 + \left(1 + \frac{1}{n}\right) + \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}$$

$$\int_1^2 (x^2 + 1)\,dx = \lim_{n\to\infty}\left(\frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}\right) = \frac{10}{3}$$

8. $y = 3x^2 + 2$ on $[-1, 2]$. $\left(\textbf{Note: } \Delta x = \dfrac{2-(-1)}{n} = \dfrac{3}{n}; \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(-1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \frac{3}{n}\sum_{i=1}^{n}\left[3\left(-1 + \frac{3i}{n}\right)^2 + 2\right.$$

$$= \frac{3}{n}\sum_{i=1}^{n}\left[3\left(1 - \frac{6i}{n} + \frac{9i^2}{n^2}\right) + 2\right]$$

$$= \frac{3}{n}\left[3n - \frac{18}{n}\frac{n(n+1)}{2} + \frac{27}{n^2}\frac{n(n+1)(2n+1)}{6} + 2n\right]$$

$$= 15 - \frac{27(n+1)}{n} + \frac{27}{2}\frac{(n+1)(2n+1)}{n^2}$$

$$\int_{-1}^{2}(3x^2 + 2)dx = \lim_{n\to\infty}\left[15 - 27\frac{(n+1)}{n} + \frac{27}{2}\frac{(n+1)(2n+1)}{n^2}\right]$$

$$= 15 - 27 + 27 = 15$$

9. $\displaystyle\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(3c_i + 10)\,\Delta x_i = \int_{-1}^{5}(3x + 10)\,dx$

on the interval $[-1, 5]$.

10. $\displaystyle\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 6c_i(4 - c_i)^2\,\Delta x_i = \int_0^4 6x(4 - x)^2\,dx$

on the interval $[0, 4]$.

11. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{c_i^2 + 4}\, \Delta x_i = \int_0^3 \sqrt{x^2 + 4}\, dx$

on the interval $[0, 3]$.

12. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \left(\frac{3}{c_i^2}\right) \Delta x_i = \int_1^3 \frac{3}{x^2}\, dx$

on the interval $[1, 3]$.

13. $\int_0^5 3\, dx$

14. $\int_0^2 (4 - 2x)\, dx$

15. $\int_{-4}^4 (4 - |x|)\, dx$

16. $\int_0^2 x^2\, dx$

17. $\int_{-2}^2 (4 - x^2)\, dx$

18. $\int_{-1}^1 \frac{1}{x^2 + 1}\, dx$

19. $\int_0^{\pi} \sin x\, dx$

20. $\int_0^{\pi/4} \tan x\, dx$

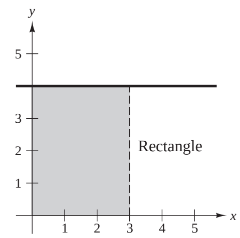
21. $\int_0^2 y^3\, dy$

22. $\int_0^2 (y - 2)^2\, dy$

23. Rectangle

$A = bh = 3(4)$

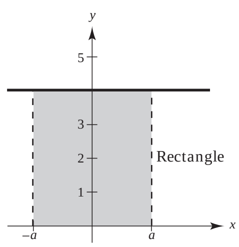
$A = \int_0^3 4\, dx = 12$



24. Rectangle

$A = bh = 2(4)(a)$

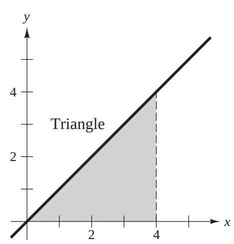
$A = \int_{-a}^{a} 4\, dx = 8a$



25. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(4)(4)$

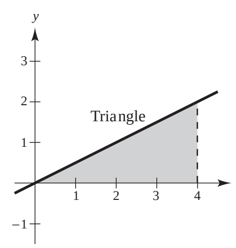
$A = \int_0^4 x\, dx = 8$



26. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(4)(2)$

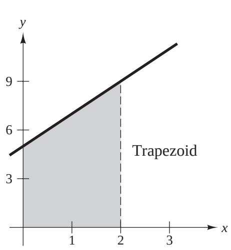
$A = \int_0^4 \frac{x}{2}\, dx = 4$



27. Trapezoid

$A = \frac{b_1 + b_2}{2}h = \left(\frac{5 + 9}{2}\right)2$

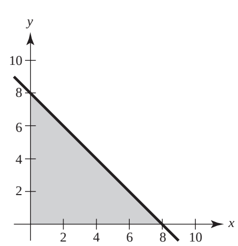
$A = \int_0^2 (2x + 5)\, dx = 14$



28. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(8)(8) = 32$

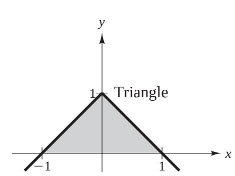
$A = \int_0^8 (8 - x) dx = 32$



29. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(2)(1)$

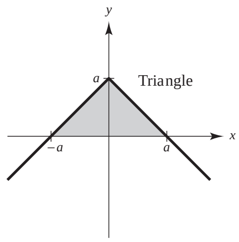
$A = \int_{-1}^1 (1 - |x|)\, dx = 1$



30. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(2a)a$

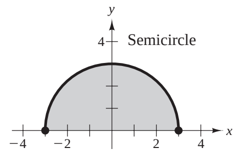
$A = \int_{-a}^{a} (a - |x|)\, dx = a^2$



31. Semicircle

$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(3)^2$

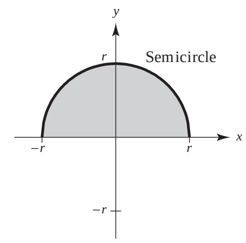
$A = \int_{-3}^{3} \sqrt{9 - x^2}\, dx = \frac{9\pi}{2}$



32. Semicircle

$A = \frac{1}{2}\pi r^2$

$A = \int_{-r}^{r} \sqrt{r^2 - x^2}\, dx = \frac{1}{2}\pi r^2$



In Exercises 33–40, $\int_2^4 x^3\, dx = 60,\ \int_2^4 x\, dx = 6,\ \int_2^4 dx = 2$

33. $\int_4^2 x\, dx = -\int_2^4 x\, dx = -6$

34. $\int_2^2 x^3\, dx = 0$

35. $\int_2^4 4x\, dx = 4\int_2^4 x\, dx = 4(6) = 24$

36. $\int_2^4 15\, dx = 15\int_2^4 dx = 15(2) = 30$

37. $\int_2^4 (x - 8)\, dx = \int_2^4 x\, dx - 8\int_2^4 dx = 6 - 8(2) = -10$

38. $\int_2^4 (x^3 + 4)dx = \int_2^4 x^3\, dx + 4\int_2^4 dx = 60 + 4(2) = 68$

39. $\int_2^4 \left(\frac{1}{2}x^3 - 3x + 2\right) dx = \frac{1}{2}\int_2^4 x^3\, dx - 3\int_2^4 x\, dx + 2\int_2^4 dx$

$= \frac{1}{2}(60) - 3(6) + 2(2) = 16$

40. $\int_2^4 (6 + 2x - x^3)dx = 6\int_2^4 dx + 2\int_2^4 x\, dx - \int_2^4 x^3\, dx$

$= 6(2) + 2(6) - 60 = -36$

41. (a) $\int_0^7 f(x)\, dx = \int_0^5 f(x)\, dx + \int_5^7 f(x)\, dx = 10 + 3 = 13$

(b) $\int_5^0 f(x)\, dx = -\int_0^5 f(x)\, dx = -10$

(c) $\int_5^5 f(x)\, dx = 0$

(d) $\int_0^5 3f(x)\, dx = 3\int_0^5 f(x)\, dx = 3(10) = 30$

42. (a) $\int_0^6 f(x)\, dx = \int_0^3 f(x)\, dx + \int_3^6 f(x)\, dx = 4 + (-1) = 3$

(b) $\int_6^3 f(x)\, dx = -\int_3^6 f(x)\, dx = -(-1) = 1$

(c) $\int_3^3 f(x)\, dx = 0$

(d) $\int_3^6 -5f(x)\, dx = -5\int_3^6 f(x)\, dx = -5(-1) = 5$

43. (a) $\int_2^6 [f(x) + g(x)]\, dx = \int_2^6 f(x)\, dx + \int_2^6 g(x)\, dx$

$= 10 + (-2) = 8$

(b) $\int_2^6 [g(x) - f(x)]\, dx = \int_2^6 g(x)\, dx - \int_2^6 f(x)\, dx$

$= -2 - 10 = -12$

(c) $\int_2^6 2g(x)\, dx = 2\int_2^6 g(x)\, dx = 2(-2) = -4$

(d) $\int_2^6 3f(x)\, dx = 3\int_2^6 f(x)\, dx = 3(10) = 30$

44. (a) $\int_{-1}^0 f(x)\, dx = \int_{-1}^1 f(x)\, dx - \int_0^1 f(x)\, dx = 0 - 5 = -5$

(b) $\int_0^1 f(x)\, dx - \int_1^0 f(x)\, dx = 5 - (-5) = 10$

(c) $\int_{-1}^1 3f(x)\, dx = 3\int_{-1}^1 f(x)\, dx = 3(0) = 0$

(d) $\int_0^1 3f(x)\, dx = 3\int_0^1 f(x)\, dx = 3(5) = 15$

45. Lower estimate: $[24 + 12 - 4 - 20 - 36](2) = -48$

Upper estimate: $[32 + 24 + 12 - 4 - 20](2) = 88$

46. (a) $[-6 + 8 + 30](2) = 64$ left endpoint estimate

(b) $[8 + 30 + 80](2) = 236$ right endpoint estimate

(c) $[0 + 18 + 50](2) = 136$ midpoint estimate

If f is increasing, then (a) is below the actual value and (b) is above.

47. (a) Quarter circle below x-axis: $-\frac{1}{4}\pi r^2 = -\frac{1}{4}\pi(2)^2 = -\pi$

(b) Triangle: $\frac{1}{2}bh = \frac{1}{2}(4)(2) = 4$

(c) Triangle + Semicircle below x-axis: $-\frac{1}{2}(2)(1) - \frac{1}{2}\pi(2)^2 = -(1 + 2\pi)$

(d) Sum of parts (b) and (c): $4 - (1 + 2\pi) = 3 - 2\pi$

(e) Sum of absolute values of (b) and (c): $4 + (1 + 2\pi) = 5 + 2\pi$

(f) Answer to (d) plus $2(10) = 20$: $(3 - 2\pi) + 20 = 23 - 2\pi$

48. (a) $\int_0^1 -f(x)\,dx = -\int_0^1 f(x)dx = \frac{1}{2}$

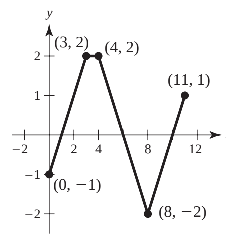
(b) $\int_3^4 3f(x)\,dx = 3(2) = 6$

(c) $\int_0^7 f(x)\,dx = -\frac{1}{2} + \frac{1}{2}(2)(2) + 2 + \frac{1}{2}(2)(2) - \frac{1}{2} = 5$

(d) $\int_5^{11} f(x)\,dx = \frac{1}{2} - \frac{1}{2}(4)(2) + \frac{1}{2} = -3$

(e) $\int_0^{11} f(x)\,dx = -\frac{1}{2} + 2 + 2 + 2 - 4 + \frac{1}{2} = 2$

(f) $\int_4^{10} f(x)\,dx = 2 - 4 = -2$



49. (a) $\int_0^5 [f(x) + 2]\,dx = \int_0^5 f(x)\,dx + \int_0^5 2\,dx = 4 + 10 = 14$

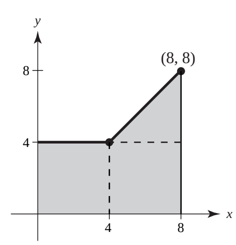
(b) $\int_{-2}^{3} f(x + 2)\,dx = \int_0^5 f(x)\,dx = 4$ (Let $u = x + 2$.)

(c) $\int_{-5}^{5} f(x)\,dx = 2\int_0^5 f(x)\,dx = 2(4) = 8$ (f even)

(d) $\int_{-5}^{5} f(x)\,dx = 0$ (f odd)

50. $f(x) = \begin{cases} 4, & x < 4 \\ x, & x \geq 4 \end{cases}$

$\int_0^8 f(x)\,dx = 4(4) + 4(4) + \frac{1}{2}(4)(4) = 40$



51. The left endpoint approximation will be greater than the actual area: $>$

52. The right endpoint approximation will be less than the actual area: $<$

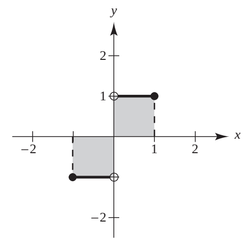
53. $f(x) = \dfrac{1}{x - 4}$

is not integrable on the interval $[3, 5]$ because f has a discontinuity at $x = 4$.

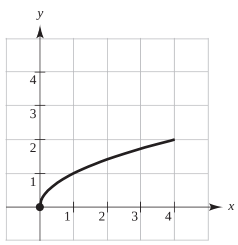
54. $f(x) = |x|/x$ is integrable on $[-1, 1]$, but is not continuous on $[-1, 1]$. There is discontinuity at $x = 0$. To see that

$$\int_{-1}^{1} \frac{|x|}{x}\, dx$$

is integrable, sketch a graph of the region bounded by $f(x) = |x|/x$ and the x-axis for $-1 \le x \le 1$. You see that the integral equals 0.

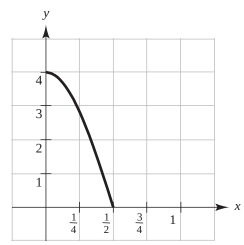


55.



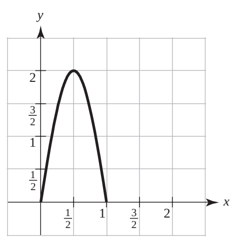
(a) $A \approx 5$ square units

56.



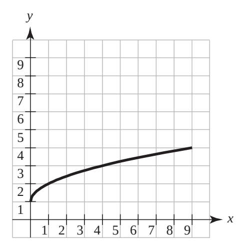
(b) $A \approx \frac{4}{3}$ square units

57.



(d) $\displaystyle\int_0^1 2 \sin \pi x\, dx \approx \frac{1}{2}(1)(2) \approx 1$

58.



(c) Area ≈ 27.

59. $\displaystyle\int_0^3 x\sqrt{3 - x}\, dx$

n	4	8	12	16	20
$L(n)$	3.6830	3.9956	4.0707	4.1016	4.1177
$M(n)$	4.3082	4.2076	4.1838	4.1740	4.1690
$R(n)$	3.6830	3.9956	4.0707	4.1016	4.1177

60. $\displaystyle\int_0^3 \frac{5}{x^2 + 1}\, dx$

n	4	8	12	16	20
$L(n)$	7.9224	7.0855	6.8062	6.6662	6.5822
$M(n)$	6.2485	6.2470	6.2460	6.2457	6.2455
$R(n)$	4.5474	5.3980	5.6812	5.8225	5.9072

61. $\displaystyle\int_0^{\pi/2} \sin^2 x\,dx$

n	4	8	12	16	20
$L(n)$	0.5890	0.6872	0.7199	0.7363	0.7461
$M(n)$	0.7854	0.7854	0.7854	0.7854	0.7854
$R(n)$	0.9817	0.8836	0.8508	0.8345	0.8247

62. $\displaystyle\int_0^{3} x\sin x\,dx$

n	4	8	12	16	20
$L(n)$	2.8186	2.9985	3.0434	3.0631	3.0740
$M(n)$	3.1784	3.1277	3.1185	3.1152	3.1138
$R(n)$	3.1361	3.1573	3.1493	3.1425	3.1375

63. True

64. False

$$\int_0^1 x\sqrt{x}\,dx \neq \left(\int_0^1 x\,dx\right)\left(\int_0^1 \sqrt{x}\,dx\right)$$

65. True

66. True

67. False

$$\int_0^2 (-x)\,dx = -2$$

68. True. The limits of integration are the same.

69. $f(x) = x^2 + 3x,\ [0, 8]$

$x_0 = 0,\ x_1 = 1,\ x_2 = 3,\ x_3 = 7,\ x_4 = 8$

$\Delta x_1 = 1,\ \Delta x_2 = 2,\ \Delta x_3 = 4,\ \Delta x_4 = 1$

$c_1 = 1,\ c_2 = 2,\ c_3 = 5,\ c_4 = 8$

$$\sum_{i=1}^{4} f(c_i)\,\Delta x = f(1)\,\Delta x_1 + f(2)\,\Delta x_2 + f(5)\,\Delta x_3 + f(8)\,\Delta x_4$$

$$= (4)(1) + (10)(2) + (40)(4) + (88)(1) = 272$$

70. $f(x) = \sin x,\ [0, 2\pi]$

$x_0 = 0,\ x_1 = \dfrac{\pi}{4},\ x_2 = \dfrac{\pi}{3},\ x_3 = \pi,\ x_4 = 2\pi$

$\Delta x_1 = \dfrac{\pi}{4},\ \Delta x_2 = \dfrac{\pi}{12},\ \Delta x_3 = \dfrac{2\pi}{3},\ \Delta x_4 = \pi$

$c_1 = \dfrac{\pi}{6},\ c_2 = \dfrac{\pi}{3},\ c_3 = \dfrac{2\pi}{3},\ c_4 = \dfrac{3\pi}{2}$

$$\sum_{i=1}^{4} f(c_i)\,\Delta x_i = f\left(\frac{\pi}{6}\right)\Delta x_1 + f\left(\frac{\pi}{3}\right)\Delta x_2 + f\left(\frac{2\pi}{3}\right)\Delta x_3 + f\left(\frac{3\pi}{2}\right)\Delta x_4$$

$$= \left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{12}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{2\pi}{3}\right) + (-1)(\pi) \approx -0.708$$

71. $\Delta x = \dfrac{b-a}{n}$, $c_i = a + i(\Delta x) = a + i\left(\dfrac{b-a}{n}\right)$

$$\begin{aligned}\int_a^b x\,dx &= \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[a + i\left(\frac{b-a}{n}\right)\right]\left(\frac{b-a}{n}\right) \\ &= \lim_{n\to\infty} \left[\left(\frac{b-a}{n}\right)\sum_{i=1}^{n} a + \left(\frac{b-a}{n}\right)^2 \sum_{i=1}^{n} i\right] \\ &= \lim_{n\to\infty} \left[\frac{b-a}{n}(an) + \left(\frac{b-a}{n}\right)^2 \frac{n(n+1)}{2}\right] \\ &= \lim_{n\to\infty} \left[a(b-a) + \frac{(b-a)^2}{n}\,\frac{n+1}{2}\right] \\ &= a(b-a) + \frac{(b-a)^2}{2} \\ &= (b-a)\left[a + \frac{b-a}{2}\right] \\ &= \frac{(b-a)(a+b)}{2} = \frac{b^2-a^2}{2}\end{aligned}$$

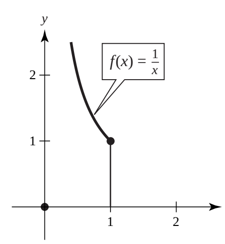
72. $\Delta x = \dfrac{b-a}{n}$, $c_i = a + i(\Delta x) = a + i\left(\dfrac{b-a}{n}\right)$

$$\begin{aligned}\int_a^b x^2\,dx &= \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[a + i\left(\frac{b-a}{n}\right)\right]^2\left(\frac{b-a}{n}\right) \\ &= \lim_{n\to\infty} \left[\left(\frac{b-a}{n}\right)\sum_{i=1}^{n}\left(a^2 + \frac{2ai(b-a)}{n} + i^2\left(\frac{b-a}{n}\right)^2\right)\right] \\ &= \lim_{n\to\infty} \left(\frac{b-a}{n}\right)\left[na^2 + \frac{2a(b-a)}{n}\,\frac{n(n+1)}{2} + \left(\frac{b-a}{n}\right)^2 \frac{n(n+1)(2n+1)}{6}\right] \\ &= \lim_{n\to\infty} \left[a^2(b-a) + \frac{a(b-a)^2(n+1)}{n} + \frac{(b-a)^3}{6}\,\frac{(n+1)(2n+1)}{n^2}\right] \\ &= a^2(b-a) + a(b-a)^2 + \frac{1}{3}(b-a)^3 \\ &= \frac{1}{3}(b^3 - a^3)\end{aligned}$$

73. $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational}\end{cases}$

is not integrable on the interval $[0, 1]$. As $\|\Delta\| \to 0$, $f(c_i) = 1$ or $f(c_i) = 0$ in each subinterval since there are an infinite number of both rational and irrational numbers in any interval, no matter how small.

74. $f(x) = \begin{cases} 0, & x = 0 \\ \dfrac{1}{x}, & 0 < x \le 1\end{cases}$

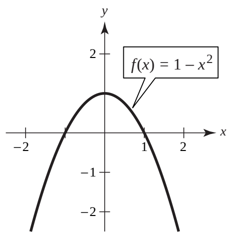


The limit

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

does not exist. This does not contradict Theorem 4.4 because f is not continuous on $[0, 1]$.

75. The function f is nonnegative between $x = -1$ and $x = 1$.

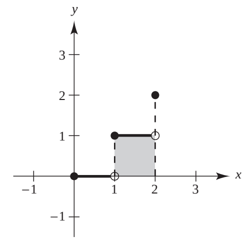


Hence,

$$\int_a^b (1 - x^2)\,dx$$

is a maximum for $a = -1$ and $b = 1$.

76. To find $\int_0^2 [\![x]\!]\,dx$, use a geometric approach.



Thus,

$$\int_0^2 [\![x]\!]\,dx = 1(2 - 1) = 1.$$

77. Let $f(x) = x^2$, $0 \le x \le 1$, and $\Delta x_i = 1/n$. The appropriate Riemann Sum is

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2.$$

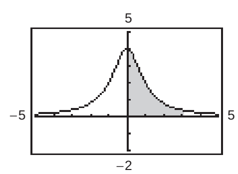
$$\lim_{n\to\infty} \frac{1}{n^3}[1^2 + 2^2 + 3^2 + \cdots + n^2] = \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n(2n + 1)(n + 1)}{6}$$

$$= \lim_{n\to\infty} \frac{2n^2 + 3n + 1}{6n^2} = \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}$$

Section 4.4 The Fundamental Theorem of Calculus

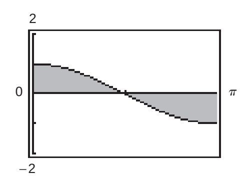
1. $f(x) = \dfrac{4}{x^2 + 1}$

$\displaystyle\int_0^{\pi} \frac{4}{x^2 + 1}\,dx$ is positive.



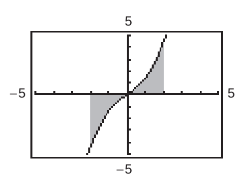
2. $f(x) = \cos x$

$$\int_0^{\pi} \cos x\,dx = 0$$



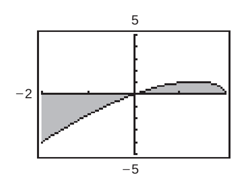
3. $f(x) = x\sqrt{x^2 + 1}$

$$\int_{-2}^{2} x\sqrt{x^2 + 1}\,dx = 0$$



4. $f(x) = x\sqrt{2 - x}$

$\displaystyle\int_{-2}^{2} x\sqrt{2 - x}\,dx$ is negative.



5. $\displaystyle\int_0^1 2x\,dx = \Big[x^2\Big]_0^1 = 1 - 0 = 1$

6. $\displaystyle\int_2^7 3\,dv = \Big[3v\Big]_2^7 = 3(7) - 3(2) = 15$

7. $\displaystyle\int_{-1}^{0} (x - 2)\,dx = \left[\frac{x^2}{2} - 2x\right]_{-1}^{0} = 0 - \left(\frac{1}{2} + 2\right) = -\frac{5}{2}$

8. $\displaystyle\int_2^5 (-3v + 4)\,v = \left[-\frac{3}{2}v^2 + 4v\right]_2^5 = \left(-\frac{75}{2} + 20\right) - (-6 + 8) = -\frac{39}{2}$

9. $\displaystyle\int_{-1}^{1} (t^2 - 2)\,dt = \left[\frac{t^3}{3} - 2t\right]_{-1}^{1} = \left(\frac{1}{3} - 2\right) - \left(-\frac{1}{3} + 2\right) = -\frac{10}{3}$

10. $\int_1^3 (3x^2 + 5x - 4)dx = \left[x^3 + \frac{5x^2}{2} - 4x\right]_1^3 = \left(27 + \frac{45}{2} - 12\right) - \left(1 + \frac{5}{2} - 4\right)$
$= 38$

11. $\int_0^1 (2t - 1)^2\, dt = \int_0^1 (4t^2 - 4t + 1)\, dt = \left[\frac{4}{3}t^3 - 2t^2 + t\right]_0^1 = \frac{4}{3} - 2 + 1 = \frac{1}{3}$

12. $\int_{-1}^1 (t^3 - 9t)\, dt = \left[\frac{1}{4}t^4 - \frac{9}{2}t^2\right]_{-1}^1 = \left(\frac{1}{4} - \frac{9}{2}\right) - \left(\frac{1}{4} - \frac{9}{2}\right) = 0$

13. $\int_1^2 \left(\frac{3}{x^2} - 1\right) dx = \left[-\frac{3}{x} - x\right]_1^2 = \left(-\frac{3}{2} - 2\right) - (-3 - 1) = \frac{1}{2}$

14. $\int_{-2}^{-1} \left(u - \frac{1}{u^2}\right) du = \left[\frac{u^2}{2} + \frac{1}{u}\right]_{-2}^{-1} = \left(\frac{1}{2} - 1\right) - \left(2 - \frac{1}{2}\right) = -2$

15. $\int_1^4 \frac{u - 2}{\sqrt{u}}\, du = \int_1^4 (u^{1/2} - 2u^{-1/2})\, du = \left[\frac{2}{3}u^{3/2} - 4u^{1/2}\right]_1^4 = \left[\frac{2}{3}\left(\sqrt{4}\right)^3 - 4\sqrt{4}\right] - \left[\frac{2}{3} - 4\right] = \frac{2}{3}$

16. $\int_{-3}^3 v^{1/3}\, dv = \left[\frac{3}{4}v^{4/3}\right]_{-3}^3 = \frac{3}{4}\left[\left(\sqrt[3]{-3}\right)^4\right] - \left(\sqrt[3]{-3}\right)^4\Big] = 0$

17. $\int_{-1}^1 \left(\sqrt[3]{t} - 2\right) dt = \left[\frac{3}{4}t^{4/3} - 2t\right]_{-1}^1 = \left(\frac{3}{4} - 2\right) - \left(\frac{3}{4} + 2\right) = -4$

18. $\int_1^8 \sqrt{\frac{2}{x}}\, dx = \sqrt{2}\int_1^8 x^{-1/2}\, dx = \left[\sqrt{2}(2)x^{1/2}\right]_1^8 = \left[2\sqrt{2x}\right]_1^8 = 8 - 2\sqrt{2}$

19. $\int_0^1 \frac{x - \sqrt{x}}{3}\, dx = \frac{1}{3}\int_0^1 (x - x^{1/2})\, dx = \frac{1}{3}\left[\frac{x^2}{2} - \frac{2}{3}x^{3/2}\right]_0^1 = \frac{1}{3}\left(\frac{1}{2} - \frac{2}{3}\right) = -\frac{1}{18}$

20. $\int_0^2 (2 - t)\sqrt{t}\, dt = \int_0^2 (2t^{1/2} - t^{3/2})\, dt = \left[\frac{4}{3}t^{3/2} - \frac{2}{5}t^{5/2}\right]_0^2 = \left[\frac{t\sqrt{t}}{15}(20 - 6t)\right]_0^2 = \frac{2\sqrt{2}}{15}(20 - 12) = \frac{16\sqrt{2}}{15}$

21. $\int_{-1}^0 (t^{1/3} - t^{2/3})\, dt = \left[\frac{3}{4}t^{4/3} - \frac{3}{5}t^{5/3}\right]_{-1}^0 = 0 - \left(\frac{3}{4} + \frac{3}{5}\right) = -\frac{27}{20}$

22. $\int_{-8}^{-1} \frac{x - x^2}{2\sqrt[3]{x}}\, dx = \frac{1}{2}\int_{-8}^{-1} (x^{2/3} - x^{5/3})\, dx$
$= \frac{1}{2}\left[\frac{3}{5}x^{5/3} - \frac{3}{8}x^{8/3}\right]_{-8}^{-1} = \left[\frac{x^{5/3}}{80}(24 - 15x)\right]_{-8}^{-1} = -\frac{1}{80}(39) + \frac{32}{80}(144) = \frac{4569}{80}$

23. $\int_0^3 |2x - 3|\, dx = \int_0^{3/2} (3 - 2x)\, dx + \int_{3/2}^3 (2x - 3)\, dx$ $\left(\text{split up the integral at the zero } x = \frac{3}{2}\right)$
$= \left[3x - x^2\right]_0^{3/2} + \left[x^2 - 3x\right]_{3/2}^3 = \left(\frac{9}{2} - \frac{9}{4}\right) - 0 + (9 - 9) - \left(\frac{9}{4} - \frac{9}{2}\right) = 2\left(\frac{9}{2} - \frac{9}{4}\right) = \frac{9}{2}$

24. $$\int_1^4 (3 - 1x - 31)\,dx = \int_1^3 [3 + (x - 3)]\,dx + \int_3^4 [3 - (x - 3)]\,dx$$
$$= \int_1^3 x\,dx + \int_3^4 (6 - x)\,dx$$
$$= \left[\frac{x^2}{2}\right]_1^3 + \left[6x - \frac{x^2}{2}\right]_3^4$$
$$= \left(\frac{9}{2} - \frac{1}{2}\right) + \left[(24 - 8) - \left(18 - \frac{9}{2}\right)\right]$$
$$= 4 + 16 - 18 + \frac{9}{2} = \frac{13}{2}$$

25. $$\int_0^3 |x^2 - 4|\,dx = \int_0^2 (4 - x^2)\,dx + \int_2^3 (x^2 - 4)\,dx$$
$$= \left[4x - \frac{x^3}{3}\right]_0^2 + \left[\frac{x^3}{3} - 4x\right]_2^3$$
$$= \left(8 - \frac{8}{3}\right) + (9 - 12) - \left(\frac{8}{3} - 8\right)$$
$$= \frac{23}{3}$$

26. $$\int_0^4 |x^2 - 4x + 3|\,dx = \int_0^1 (x^2 - 4x + 3)\,dx - \int_1^3 (x^2 - 4x + 3)\,dx + \int_3^4 (x^2 - 4x + 3)\,dx$$ (split up the integral at the zeros $x = 1, 3$)
$$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_0^1 - \left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3 + \left[\frac{x^3}{3} - 2x^2 + 3x\right]_3^4$$
$$= \left(\frac{1}{3} - 2 + 3\right) - (9 - 18 + 9) + \left(\frac{1}{3} - 2 + 3\right) + \left(\frac{64}{3} - 32 + 12\right) - (9 - 18 + 9)$$
$$= \frac{4}{3} - 0 + \frac{4}{3} + \frac{4}{3} - 0 = 4$$

27. $$\int_0^{\pi} (1 + \sin x)\,dx = \Big[x - \cos x\Big]_0^{\pi} = (\pi + 1) - (0 - 1) = 2 + \pi$$

28. $$\int_0^{\pi/4} \frac{1 - \sin^2\theta}{\cos^2\theta}\,d\theta = \int_0^{\pi/4} d\theta = \Big[\theta\Big]_0^{\pi/4} = \frac{\pi}{4}$$

29. $$\int_{-\pi/6}^{\pi/6} \sec^2 x\,dx = \Big[\tan x\Big]_{-\pi/6}^{\pi/6} = \frac{\sqrt{3}}{3} - \left(-\frac{\sqrt{3}}{3}\right) = \frac{2\sqrt{3}}{3}$$

30. $$\int_{\pi/4}^{\pi/2} (2 - \csc^2 x)\,dx = \Big[2x + \cot x\Big]_{\pi/4}^{\pi/2} = (\pi + 0) - \left(\frac{\pi}{2} + 1\right) = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$$

31. $$\int_{-\pi/3}^{\pi/3} 4\sec\theta\tan\theta\,d\theta = \Big[4\sec\theta\Big]_{-\pi/3}^{\pi/3} = 4(2) - 4(2) = 0$$

32. $$\int_{-\pi/2}^{\pi/2} (2t + \cos t)\,dt = \Big[t^2 + \sin t\Big]_{-\pi/2}^{\pi/2} = \left(\frac{\pi^2}{4} + 1\right) - \left(\frac{\pi^2}{4} - 1\right) = 2$$

33. $$A = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$$

34. $$A = \int_{-1}^1 (1 - x^4)\,dx = \left[x - \frac{1}{5}x^5\right]_{-1}^1 = \frac{8}{5}$$

35. $$A = \int_0^3 (3 - x)\sqrt{x}\,dx = \int_0^3 (3x^{1/2} - x^{3/2})\,dx = \left[2x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^3 = \left[\frac{x\sqrt{x}}{5}(10 - 2x)\right]_0^3 = \frac{12\sqrt{3}}{5}$$

36. $A = \int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$

37. $A = \int_0^{\pi/2} \cos x\,dx = \left[\sin x\right]_0^{\pi/2} = 1$

38. $A = \int_0^{\pi} (x + \sin x)\,dx = \left[\frac{x^2}{2} - \cos x\right]_0^{\pi} = \frac{\pi^2}{2} + 2 = \frac{\pi^2 + 4}{2}$

39. Since $y \geq 0$ on $[0, 2]$,

$$A = \int_0^2 (3x^2 + 1)\,dx = \left[x^3 + x\right]_0^2 = 8 + 2 = 10.$$

40. Since $y \geq 0$ on $[0. 8]$,

$$\text{Area} = \int_0^8 (1 + x^{1/3})\,dx = \left[x + \frac{3}{4}x^{4/3}\right]_0^8 = 8 + \frac{3}{4}(16) = 20.$$

41. Since $y \geq 0$ on $[0, 2]$,

$$A = \int_0^2 (x^3 + x)\,dx = \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^2 = 4 + 2 = 6.$$

42. Since $y \geq 0$ on $[0, 3]$,

$$A = \int_0^3 (3x - x^2)\,dx = \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_0^3 = \frac{9}{2}.$$

43. $\int_0^2 \left(x - 2\sqrt{x}\right)\,dx = \left[\frac{x^2}{2} - \frac{4x^{3/2}}{3}\right]_0^2 = 2 - \frac{8\sqrt{2}}{3}$

$$f(c)(2 - 0) = \frac{6 - 8\sqrt{2}}{3}$$

$$c - 2\sqrt{c} = \frac{3 - 4\sqrt{2}}{3}$$

$$c - 2\sqrt{c} + 1 = \frac{3 - 4\sqrt{2}}{3} + 1$$

$$\left(\sqrt{c} - 1\right)^2 = \frac{6 - 4\sqrt{2}}{3}$$

$$\sqrt{c} - 1 = \pm\sqrt{\frac{6 - 4\sqrt{2}}{3}}$$

$$c = \left[1 \pm \sqrt{\frac{6 - 4\sqrt{2}}{3}}\right]^2$$

$$c \approx 0.4380 \text{ or } c \approx 1.7908$$

44. $\int_1^3 \frac{9}{x^3}\,dx = \left[-\frac{9}{2x^2}\right]_1^3 = -\frac{1}{2} + \frac{9}{2} = 4$

$$f(c)(3 - 1) = 4$$

$$\frac{9}{c^3} = 2$$

$$c^3 = \frac{9}{2}$$

$$c = \sqrt[3]{\frac{9}{2}} \approx 1.6510$$

45. $\int_{-\pi/4}^{\pi/4} 2\sec^2 x\,dx = \left[2\tan x\right]_{-\pi/4}^{\pi/4} = 2(1) - 2(-1) = 4$

$$f(c)\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = 4$$

$$2\sec^2 c = \frac{8}{\pi}$$

$$\sec^2 c = \frac{4}{\pi}$$

$$\sec c = \pm\frac{2}{\sqrt{\pi}}$$

$$c = \pm\text{arcsec}\left(\frac{2}{\sqrt{\pi}}\right)$$

$$= \pm\arccos\frac{\sqrt{\pi}}{2} \approx \pm 0.4817$$

46. $\int_{-\pi/3}^{\pi/3} \cos x\,dx = \left[\sin x\right]_{-\pi/3}^{\pi/3} = \sqrt{3}$

$$f(c)\left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right] = \sqrt{3}$$

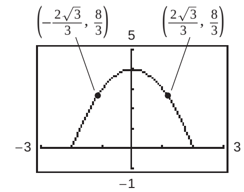
$$\cos c = \frac{3\sqrt{3}}{2\pi}$$

$$c \approx \pm 0.5971$$

47. $\frac{1}{2-(-2)}\int_{-2}^{2}(4-x^2)\,dx = \frac{1}{4}\left[4x-\frac{1}{3}x^3\right]_{-2}^{2} = \frac{1}{4}\left[\left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)\right] = \frac{8}{3}$

Average value $= \frac{8}{3}$

$4-x^2 = \frac{8}{3}$ when $x^2 = 4-\frac{8}{3}$ or $x = \pm\frac{2\sqrt{3}}{3} \approx \pm 1.155$.



48. $\frac{1}{3-1}\int_1^3 \frac{4(x^2+1)}{x^2}\,dx = 2\int_1^3 (1+x^{-2})dx = 2\left[x-\frac{1}{x}\right]_1^3$

$$= 2\left(3-\frac{1}{3}\right) = \frac{16}{3}$$

Average value $= \frac{16}{3}$

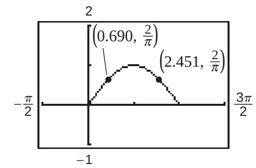
$\frac{4(x^2+1)}{x^2} = \frac{16}{3} \implies x = \pm\sqrt{3}$

49. $\frac{1}{\pi-0}\int_0^{\pi}\sin x\,dx = \left[-\frac{1}{\pi}\cos x\right]_0^{\pi} = \frac{2}{\pi}$

Average value $= \frac{2}{\pi}$

$\sin x = \frac{2}{\pi}$

$x \approx 0.690, 2.451$

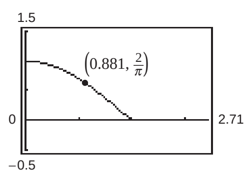


50. $\frac{1}{(\pi/2)-0}\int_0^{\pi/2}\cos x\,dx = \left[\frac{2}{\pi}\sin x\right]_0^{\pi/2} = \frac{2}{\pi}$

Average value $= \frac{2}{\pi}$

$\cos x = \frac{2}{\pi}$

$x \approx 0.881$



51. The distance traveled is $\int_0^8 v(t)\,dt$. The area under the curve from $0 \le t \le 8$ is approximately (18 squares) (30) $\approx$ 540 ft.

52. The distance traveled is $\int_0^5 v(t)\,dt$. The area under the curve from $0 \le t \le 5$ is approximately (29 squares) (5) $=$ 145 ft.

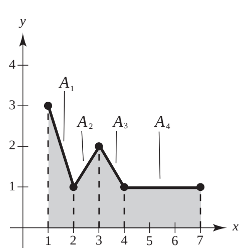
53. If f is continuous on $[a, b]$ and $F'(x) = f(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

54. (a) $\int_1^7 f(x)\,dx =$ Sum of the areas

$$= A_1 + A_2 + A_3 + A_4$$

$$= \frac{1}{2}(3+1) + \frac{1}{2}(1+2) + \frac{1}{2}(2+1) + (3)(1)$$

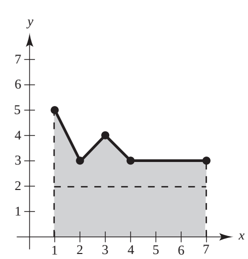
$$= 8$$



(b) Average value $= \frac{\int_1^7 f(x)\,dx}{7-1} = \frac{8}{6} = \frac{4}{3}$

(c) $A = 8 + (6)(2) = 20$

Average value $= \frac{20}{6} = \frac{10}{3}$



55. $\int_0^2 f(x)\,dx = -(\text{area of region } A) = -1.5$

56. $\int_2^6 f(x)\,dx = (\text{area or region } B) = \int_0^6 f(x)\,dx - \int_0^2 f(x)\,dx$

$$= 3.5 - (-1.5) = 5.0$$

57. $\displaystyle\int_0^6 |f(x)|\,dx = -\int_0^2 f(x)\,dx + \int_2^6 f(x)\,dx = 1.5 + 5.0 = 6.5$

58. $\displaystyle\int_0^2 -2f(x)\,dx = -2\int_0^2 f(x)\,dx = -2(-1.5) = 3.0$

59. $\displaystyle\int_0^6 [2 + f(x)]\,dx = \int_0^6 2\,dx + \int_0^6 f(x)\,dx$

$= 12 + 3.5 = 15.5$

60. Average value $\displaystyle = \frac{1}{6}\int_0^6 f(x)\,dx = \frac{1}{6}(3.5) = 0.5833$

61. (a) $F(x) = k\sec^2 x$

$F(0) = k = 500$

$F(x) = 500\sec^2 x$

(b) $\displaystyle\frac{1}{\pi/3 - 0}\int_0^{\pi/3} 500\sec^2 x\,dx = \frac{1500}{\pi}\Big[\tan x\Big]_0^{\pi/3}$

$\displaystyle = \frac{1500}{\pi}\left(\sqrt{3} - 0\right)$

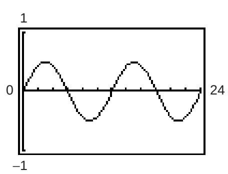
≈ 826.99 newtons

≈ 827 newtons

62. $\displaystyle\frac{1}{R - 0}\int_0^R k(R^2 - r^2)\,dr = \frac{k}{R}\left[R^2 r - \frac{r^3}{3}\right]_0^R = \frac{2kR^2}{3}$

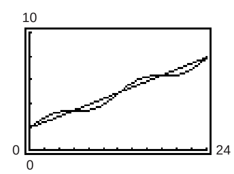
63. $\displaystyle\frac{1}{5 - 0}\int_0^5 (0.1729t + 0.1522t^2 - 0.0374t^3)\,dt \approx \frac{1}{5}\Big[0.08645t^2 + 0.05073t^3 - 0.00935t^4\Big]_0^5 \approx 0.5318$ liter

64. (a)



The area above the x-axis equals the area below the x-axis. Thus, the average value is zero.

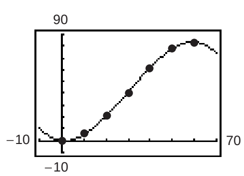
(b)



The average value of S appears to be g.

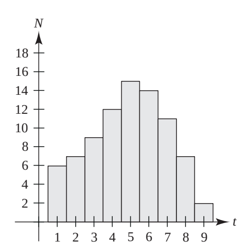
65. (a) $v = -8.61 \times 10^{-4}t^3 + 0.0782t^2 - 0.208t + 0.0952$

(b)



(c) $\displaystyle\int_0^{60} v(t)\,dt = \left[\frac{-8.61 \times 10^{-4}t^4}{4} + \frac{0.0782t^3}{3} - \frac{0.208t^2}{2} + 0.0952t\right]_0^{60} \approx 2476$ meters

66. (a) histogram



(b) $[6 + 7 + 9 + 12 + 15 + 14 + 11 + 7 + 2]60 = (83)60 = 4980$ customers

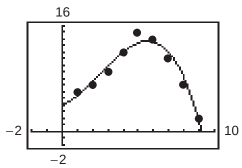
—CONTINUED—

66. —CONTINUED—

(c) Using a graphing utility, you obtain

$N(t) = -0.084175t^3 + 0.63492t^2 + 0.79052 + 4.10317.$

(d)



(e) $\int_0^9 N(t)\,dt \approx 85.162$

The estimated number of customers is $(85.162)(60) \approx 5110$.

(f) Between 3 P.M. and 7 P.M., the number of customers is approximately $\left(\int_3^7 N(t)\,dt\right)(60) \approx (50.28)(60) \approx 3017.$

Hence, $3017/240 \approx 12.6$ per minute.

67. $F(x) = \int_0^x (t - 5)\,dt = \left[\frac{t^2}{2} - 5t\right]_0^x = \frac{x^2}{2} - 5x$

$F(2) = \frac{4}{2} - 5(2) = -8$

$F(5) = \frac{25}{2} - 5(5) = -\frac{25}{2}$

$F(8) = \frac{64}{2} - 5(8) = -8$

68. $F(x) = \int_2^x (t^3 + 2t - 2)\,dt = \left[\frac{t^4}{4} + t^2 - 2t\right]_2^x$

$= \left(\frac{x^4}{4} + x^2 - 2x\right) - (4 + 4 - 4)$

$= \frac{x^4}{4} + x^2 - 2x - 4$

$F(2) = 4 + 4 - 4 - 4 = 0$ $\left[\textbf{Note: } F(2) = \int_2^2 (t^3 + 2t - 2)\,dt = 0\right]$

$F(5) = \frac{625}{4} + 25 - 10 - 4 = 167.25$

$F(8) = \frac{8^4}{4} + 64 - 16 - 4 = 1068$

69. $F(x) = \int_1^x \frac{10}{v^2}\,dv = \int_1^x 10v^{-2}\,dv = \frac{-10}{v}\Bigg]_1^x$

$= -\frac{10}{x} + 10 = 10\left(1 - \frac{1}{x}\right)$

$F(2) = 10\left(\frac{1}{2}\right) = 5$

$F(5) = 10\left(\frac{4}{5}\right) = 8$

$F(8) = 10\left(\frac{7}{8}\right) = \frac{35}{4}$

70. $F(x) = \int_2^x \frac{-2}{t^3}\,dt = -\int_2^x 2t^{-3}\,dt = \frac{1}{t^2}\Bigg]_2^x = \frac{1}{x^2} - \frac{1}{4}$

$F(2) = \frac{1}{4} - \frac{1}{4} = 0$

$F(5) = \frac{1}{25} - \frac{1}{4} = -\frac{21}{100} = -0.21$

$F(8) = \frac{1}{64} - \frac{1}{4} = -\frac{15}{64}$

71. $F(x) = \int_1^x \cos\theta\, d\theta = \sin\theta \Big]_1^x = \sin x - \sin 1$

$F(2) = \sin 2 - \sin 1 = 0.0678$

$F(5) = \sin 5 - \sin 1 \approx -1.8004$

$F(8) = \sin 8 - \sin 1 \approx 0.1479$

72. $F(x) = \int_0^x \sin\theta\, d\theta = -\cos\theta \Big]_0^x = -\cos x + \cos 0 = 1 - \cos x$

$F(2) = 1 - \cos 2 \approx 1.4161$

$F(5) = 1 - \cos 5 \approx 0.7163$

$F(8) = 1 - \cos 8 \approx 1.1455$

73. $g(x) = \int_0^x f(t)\, dt$

(a) $g(0) = \int_0^0 f(t)\, dt = 0$

$g(2) = \int_0^2 f(t)\, dt \approx 4 + 2 + 1 = 7$

$g(4) = \int_0^4 f(t)\, dt \approx 7 + 2 = 9$

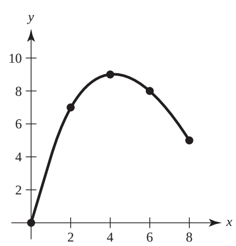
$g(6) = \int_0^6 f(t)\, dt \approx 9 + (-1) = 8$

$g(8) = \int_0^8 f(t)\, dt \approx 8 - 3 = 5$

(b) g increasing on $(0, 4)$ and decreasing on $(4, 8)$

(c) g is a maximum of 9 at $x = 4$.

(d)



74. $g(x) = \int_0^x f(t)dt$

(a) $g(0) = \int_0^0 f(t)\, dt = 0$

$g(2) = \int_0^2 f(t)\, dt = -\tfrac{1}{2}(2)(4) = -4$

$g(4) = \int_0^4 f(t)\, dt = -\tfrac{1}{2}(4)(4) = -8$

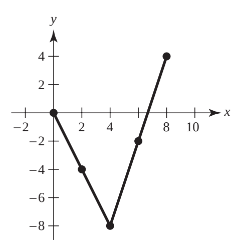
$g(6) = \int_0^6 f(t)\, dt = -8 + 2 + 4 = -2$

$g(8) = \int_0^8 f(t)\, dt = -2 + 6 = 4$

(b) g decreasing on $(0, 4)$ and increasing on $(4, 8)$

(c) g is a minimum of -8 at $x = 4$.

(d)



75. (a) $\int_0^x (t + 2)\, dt = \left[\frac{t^2}{2} + 2t\right]_0^x = \frac{1}{2}x^2 + 2x$

(b) $\frac{d}{dx}\left[\frac{1}{2}x^2 + 2x\right] = x + 2$

76. (a) $\int_0^x t(t^2 + 1)\, dt = \int_0^x (t^3 + t)\, dt = \left[\frac{1}{4}t^4 + \frac{1}{2}t^2\right]_0^x = \frac{1}{4}x^4 + \frac{1}{2}x^2 = \frac{x^2}{4}(x^2 + 2)$

(b) $\frac{d}{dx}\left[\frac{1}{4}x^4 + \frac{1}{2}x^2\right] = x^3 + x = x(x^2 + 1)$

77. (a) $\int_8^x \sqrt[3]{t}\, dt = \left[\frac{3}{4}t^{4/3}\right]_8^x = \frac{3}{4}(x^{4/3} - 16) = \frac{3}{4}x^{4/3} - 12$

(b) $\frac{d}{dx}\left[\frac{3}{4}x^{4/3} - 12\right] = x^{1/3} = \sqrt[3]{x}$

78. (a) $\int_4^x \sqrt{t}\, dt = \left[\frac{2}{3}t^{3/2}\right]_4^x = \frac{2}{3}x^{3/2} - \frac{16}{3} = \frac{2}{3}(x^{3/2} - 8)$

(b) $\frac{d}{dx}\left[\frac{2}{3}x^{3/2} - \frac{16}{3}\right] = x^{1/2} = \sqrt{x}$

79. (a) $\displaystyle\int_{x/4}^{x} \sec^2 t\, dt = \Big[\tan t\Big]_{x/4}^{x} = \tan x - 1$

(b) $\displaystyle\frac{d}{dx}[\tan x - 1] = \sec^2 x$

80. (a) $\displaystyle\int_{\pi/3}^{x} \sec t \tan t\, dt = \Big[\sec t\Big]_{\pi/3}^{x} = \sec x - 2$

(b) $\displaystyle\frac{d}{dx}[\sec x - 2] = \sec x \tan x$

81. $\displaystyle F(x) = \int_{-2}^{x} (t^2 - 2t)\, dt$

$F'(x) = x^2 - 2x$

82. $\displaystyle F(x) = \int_{1}^{x} \frac{t^2}{t^2 + 1}\, dt$

$\displaystyle F'(x) = \frac{x^2}{x^2 + 1}$

83. $\displaystyle F(x) = \int_{-1}^{x} \sqrt{t^4 + 1}\, dt$

$F'(x) = \sqrt{x^4 + 1}$

84. $\displaystyle F(x) = \int_{1}^{x} \sqrt[4]{t}\, dt$

$F'(x) = \sqrt[4]{x}$

85. $\displaystyle F(x) = \int_{0}^{x} t \cos t\, dt$

$F'(x) = x \cos x$

86. $\displaystyle F(x) = \int_{0}^{x} \sec^3 t\, dt$

$F'(x) = \sec^3 x$

87.
$$\begin{aligned} F(x) &= \int_{x}^{x+2} (4t + 1)\, dt \\ &= \Big[2t^2 + t\Big]_{x}^{x+2} \\ &= [2(x + 2)^2 + (x + 2)] - [2x^2 + x] \\ &= 8x + 10 \end{aligned}$$
$F'(x) = 8$

Alternate solution:
$$\begin{aligned} F(x) &= \int_{x}^{x+2} (4t + 1)\, dt \\ &= \int_{x}^{0} (4t + 1)\, dt + \int_{0}^{x+2} (4t + 1)\, dt \\ &= -\int_{0}^{x} (4t + 1)\, dt + \int_{0}^{x+2} (4t + 1)\, dt \end{aligned}$$
$F'(x) = -(4x + 1) + 4(x + 2) + 1 = 8$

88. $\displaystyle F(x) = \int_{-x}^{x} t^3\, dt = \left[\frac{t^4}{4}\right]_{-x}^{x} = 0$

$F'(x) = 0$

Alternate solution:
$$\begin{aligned} F(x) &= \int_{-x}^{x} t^3\, dt \\ &= \int_{-x}^{0} t^3\, dt + \int_{0}^{x} t^3\, dt \\ &= -\int_{0}^{-x} t^3\, dt + \int_{0}^{x} t^3\, dt \end{aligned}$$
$F'(x) = -(-x)^3(-1) + (x^3) = 0$

89. $\displaystyle F(x) = \int_{0}^{\sin x} \sqrt{t}\, dt = \left[\frac{2}{3}t^{3/2}\right]_{0}^{\sin x} = \frac{2}{3}(\sin x)^{3/2}$

$F'(x) = (\sin x)^{1/2} \cos x = \cos x\sqrt{\sin x}$

Alternate solution:

$\displaystyle F(x) = \int_{0}^{\sin x} \sqrt{t}\, dt$

$\displaystyle F'(x) = \sqrt{\sin x}\,\frac{d}{dx}(\sin x) = \sqrt{\sin x}(\cos x)$

90. $\displaystyle F(x) = \int_{2}^{x^2} t^{-3}\, dt = \left[\frac{t^{-2}}{-2}\right]_{2}^{x^2} = \left[-\frac{1}{2t^2}\right]_{2}^{x^2} = \frac{-1}{2x^4} + \frac{1}{8} \implies F'(x) = 2x^{-5}$

Alternate solution: $F'(x) = (x^2)^{-3}(2x) = 2x^{-5}$

91. $\displaystyle F(x) = \int_{0}^{x^3} \sin t^2\, dt$

$F'(x) = \sin(x^3)^2 \cdot 3x^2 = 3x^2 \sin x^6$

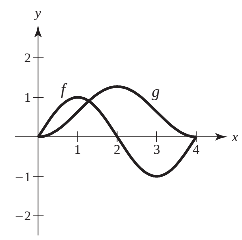
92. $\displaystyle F(x) = \int_{0}^{x^2} \sin \theta^2\, d\theta$

$F'(x) = \sin (x^2)^2 (2x) = 2x \sin x^4$

93. $g(x) = \int_0^x f(t)\,dt$

$g(0) = 0,\ g(1) \approx \frac{1}{2},\ g(2) \approx 1,\ g(3) \approx \frac{1}{2},\ g(4) = 0$

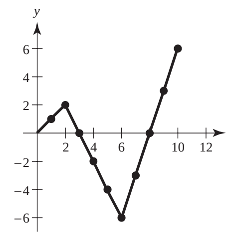
g has a relative maximum at $x = 2$.



94. (a)

x	1	2	3	4	5	6	7	8	9	10
$g(x)$	1	2	0	−2	−4	−6	−3	0	3	6

(b)



(c) Minimum of g at $(6, -6)$.

(d) Minimum at $(10, 6)$. Relative maximum at $(2, 2)$.

(e) On $[6, 10]$, g increases at a rate of $\frac{12}{4} = 3$.

(f) Zeros of g: $x = 3,\ x = 8$

95. (a) $C(x) = 5000\left(25 + 3\int_0^x t^{1/4}\,dt\right)$

$$= 5000\left(25 + 3\left[\frac{4}{5}t^{5/4}\right]_0^x\right)$$

$$= 5000\left(25 + \frac{12}{5}x^{5/4}\right) = 1000(125 + 12x^{5/4})$$

(b) $C(1) = 1000(125 + 12(1)) = \$137{,}000$

$C(5) = 1000(125 + 12(5)^{5/4}) \approx \$214{,}721$

$C(10) = 1000(125 + 12(10)^{5/4}) \approx \$338{,}394$

96. (a) $g(t) = 4 - \frac{4}{t^2}$

$\lim_{t\to\infty} g(t) = 4$

Horizontal asymptote: $y = 4$

(b) $A(x) = \int_1^x \left(4 - \frac{4}{t^2}\right) dt$

$$= \left[4t + \frac{4}{t}\right]_1^x = 4x + \frac{4}{x} - 8$$

$$= \frac{4x^2 - 8x + 4}{x} = \frac{4(x-1)^2}{x}$$

$\lim_{x\to\infty} A(x) = \lim_{x\to\infty}\left(4x + \frac{4}{x} - 8\right) = \infty + 0 - 8 = \infty$

The graph of $A(x)$ does not have a horizontal asymptote.

97. $x(t) = t^3 - 6t^2 + 9t - 2$

$x'(t) = 3t^2 - 12t + 9$

$= 3(t^2 - 4t + 3)$

$= 3(t-3)(t-1)$

Total distance $= \int_0^5 |x'(t)|\,dt$

$$= \int_0^5 3|(t-3)(t-1)|\,dt$$

$$= 3\int_0^1 (t^2 - 4t + 3)\,dt - 3\int_1^3 (t^2 - 4t + 3)\,dt + 3\int_3^5 (t^2 - 4t + 3)\,dt$$

$= 4 + 4 + 20$

$= 28$ units

98. $x(t) = (t - 1)(t - 3)^2 = t^3 - 7t^2 + 15t - 9$

$x'(t) = 3t^2 - 14t + 15$

Using a graphing utility,

Total distance $= \int_0^5 |x'(t)|\,dt \approx 27.37$ units.

99. Total distance $= \int_1^4 |x'(t)|\,dt$

$$= \int_1^4 |v(t)|\,dt$$

$$= \int_1^4 \frac{1}{\sqrt{t}}\,dt$$

$$= 2t^{1/2}\Big]_1^4$$

$$= 2(2 - 1) = 2 \text{ units}$$

100. $P = \dfrac{2}{\pi}\displaystyle\int_0^{\pi/2} \sin\theta\,d\theta = \left[-\frac{2}{\pi}\cos\theta\right]_0^{\pi/2} = -\frac{2}{\pi}(0 - 1) = \frac{2}{\pi} \approx 63.7\%$

101. True

102. True

103. The function $f(x) = x^{-2}$ is not continuous on $[-1, 1]$.

$$\int_{-1}^1 x^{-2}\,dx = \int_{-1}^0 x^{-2}\,dx + \int_0^1 x^{-2}\,dx$$

Each of these integrals is infinite. $f(x) = x^{-2}$ has a nonremovable discontinuity at $x = 0$.

104. Let $F(t)$ be an antiderivative of $f(t)$. Then,

$$\int_{u(x)}^{v(x)} f(t)\,dt = \Big[F(t)\Big]_{u(x)}^{v(x)} = F(v(x)) - F(u(x))$$

$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\,dt\right] = \frac{d}{dx}[F(v(x)) - F(u(x))]$$

$$= F'(v(x))v'(x) - F'(u(x))u'(x)$$

$$= f(v(x))v'(x) - f(u(x))u'(x).$$

105. $f(x) = \displaystyle\int_0^{1/x} \frac{1}{t^2 + 1}\,dt + \int_0^x \frac{1}{t^2 + 1}\,dt$

By the Second Fundamental Theorem of Calculus, we have

$$f'(x) = \frac{1}{(1/x)^2 + 1}\left(-\frac{1}{x^2}\right) + \frac{1}{x^2 + 1}$$

$$= -\frac{1}{1 + x^2} + \frac{1}{x^2 + 1} = 0.$$

Since $f'(x) = 0$, $f(x)$ must be constant.

106. $G(x) = \displaystyle\int_0^x \left[s\int_0^s f(t)\,dt\right] ds$

(a) $G(0) = \displaystyle\int_0^0 \left[s\int_0^s f(t)\,dt\right] ds = 0$

(b) Let $F(s) = s\displaystyle\int_0^s f(t)\,dt$.

$$G(x) = \int_0^x F(s)\,ds$$

$$G'(x) = F(x) = x\int_0^x f(t)\,dt$$

$$G'(0) = 0\int_0^0 f(t)\,dt = 0$$

(c) $G''(x) = x \cdot f(x) + \displaystyle\int_0^x f(t)\,dt$

(d) $G''(0) = 0 \cdot f(0) + \displaystyle\int_0^0 f(t)\,dt = 0$

Section 4.5 Integration by Substitution

$\int f(g(x))g'(x)\,dx$	$u = g(x)$	$du = g'(x)\,dx$
1. $\int (5x^2+1)^2(10x)\,dx$	$5x^2+1$	$10x\,dx$
2. $\int x^2\sqrt{x^3+1}\,dx$	x^3+1	$3x^2\,dx$
3. $\int \frac{x}{\sqrt{x^2+1}}\,dx$	x^2+1	$2x\,dx$
4. $\int \sec 2x \tan 2x\,dx$	$2x$	$2\,dx$
5. $\int \tan^2 x \sec^2 x\,dx$	$\tan x$	$\sec^2 x\,dx$
6. $\int \frac{\cos x}{\sin^2 x}\,dx$	$\sin x$	$\cos x\,dx$

7. $\int (1+2x)^4(2)\,dx = \frac{(1+2x)^5}{5} + C$

Check: $\frac{d}{dx}\left[\frac{(1+2x)^5}{5} + C\right] = 2(1+2x)^4$

8. $\int (x^2-9)^3(2x)\,dx = \frac{(x^2-9)^4}{4} + C$

Check: $\frac{d}{dx}\left[\frac{(x^2-9)^4}{4} + C\right] = \frac{4(x^2-9)^3}{4}(2x) = (x^2-9)^3(2x)$

9. $\int (9-x^2)^{1/2}(-2x)\,dx = \frac{(9-x^2)^{3/2}}{3/2} + C = \frac{2}{3}(9-x^2)^{3/2} + C$

Check: $\frac{d}{dx}\left[\frac{2}{3}(9-x^2)^{3/2} + C\right] = \frac{2}{3}\cdot\frac{3}{2}(9-x^2)^{1/2}(-2x) = \sqrt{9-x^2}(-2x)$

10. $\int (1-2x^2)^{1/3}(-4x)\,dx = \frac{3}{4}(1-2x^2)^{4/3} + C$

Check: $\frac{d}{dx}\left[\frac{3}{4}(1-2x^2)^{4/3} + C\right] = \frac{3}{4}\cdot\frac{4}{3}(1-2x^2)^{1/3}(-4x) = (1-2x^2)^{1/3}(-4x)$

11. $\int x^3(x^4+3)^2\,dx = \frac{1}{4}\int (x^4+3)^2(4x^3)\,dx = \frac{1}{4}\frac{(x^4+3)^3}{3} + C = \frac{(x^4+3)^3}{12} + C$

Check: $\frac{d}{dx}\left[\frac{(x^4+3)^3}{12} + C\right] = \frac{3(x^4+3)^2}{12}(4x^3) = (x^4+3)^2(x^3)$

12. $\int x^2(x^3+5)^4\,dx = \frac{1}{3}\int (x^3+5)^4(3x^2)\,dx = \frac{1}{3}\frac{(x^3+5)^5}{5} + C = \frac{(x^3+5)^5}{15} + C$

Check: $\frac{d}{dx}\left[\frac{(x^3+5)^5}{15} + C\right] = \frac{5(x^3+5)^4(3x^2)}{15} = (x^3+5)^4x^2$

13. $\int x^2(x^3 - 1)^4\,dx = \frac{1}{3}\int (x^3 - 1)^4(3x^2)\,dx = \frac{1}{3}\left[\frac{(x^3 - 1)^5}{5}\right] + C = \frac{(x^3 - 1)^5}{15} + C$

Check: $\frac{d}{dx}\left[\frac{(x^3 - 1)^5}{15} + C\right] = \frac{5(x^3 - 1)^4(3x^2)}{15} = x^2(x^3 - 1)^4$

14. $\int x(4x^2 + 3)^3\,dx = \frac{1}{8}\int (4x^2 + 3)^3(8x)\,dx = \frac{1}{8}\left[\frac{(4x^2 + 3)^4}{4}\right] + C = \frac{(4x^2 + 3)^4}{32} + C$

Check: $\frac{d}{dx}\left[\frac{(4x^2 + 3)^4}{32} + C\right] = \frac{4(4x^2 + 3)^3(8x)}{32} = x(4x^2 + 3)^3$

15. $\int t\sqrt{t^2 + 2}\,dt = \frac{1}{2}\int (t^2 + 2)^{1/2}(2t)\,dt = \frac{1}{2}\frac{(t^2 + 2)^{3/2}}{3/2} + C = \frac{(t^2 + 2)^{3/2}}{3} + C$

Check: $\frac{d}{dt}\left[\frac{(t^2 + 2)^{3/2}}{3} + C\right] = \frac{3/2(t^2 + 2)^{1/2}(2t)}{3} = (t^2 + 2)^{1/2}t$

16. $\int t^3\sqrt{t^4 + 5}\,dt = \frac{1}{4}\int (t^4 + 5)^{1/2}(4t^3)\,dt = \frac{1}{4}\frac{(t^4 + 5)^{3/2}}{3/2} + C = \frac{1}{6}(t^4 + 5)^{3/2} + C$

Check: $\frac{d}{dt}\left[\frac{1}{6}(t^4 + 5)^{3/2} + C\right] = \frac{1}{6}\cdot\frac{3}{2}(t^4 + 5)^{1/2}(4t^3) = (t^4 + 5)^{1/2}(t^3)$

17. $\int 5x(1 - x^2)^{1/3}\,dx = -\frac{5}{2}\int (1 - x^2)^{1/3}(-2x)\,dx = -\frac{5}{2}\cdot\frac{(1 - x^2)^{4/3}}{4/3} + C = -\frac{15}{8}(1 - x^2)^{4/3} + C$

Check: $\frac{d}{dx}\left[-\frac{15}{8}(1 - x^2)^{4/3} + C\right] = -\frac{15}{8}\cdot\frac{4}{3}(1 - x^2)^{1/3}(-2x) = 5x(1 - x^2)^{1/3} = 5x\sqrt[3]{1 - x^2}$

18. $\int u^2\sqrt{u^3 + 2}\,du = \frac{1}{3}\int (u^3 + 2)^{1/2}(3u^2)\,du = \frac{1}{3}\frac{(u^3 + 2)^{3/2}}{3/2} + C = \frac{2(u^3 + 2)^{3/2}}{9} + C$

Check: $\frac{d}{du}\left[\frac{2(u^3 + 2)^{3/2}}{9} + C\right] = \frac{2}{9}\cdot\frac{3}{2}(u^3 + 2)^{1/2}(3u^2) = (u^3 + 2)^{1/2}(u^2)$

19. $\int \frac{x}{(1 - x^2)^3}\,dx = -\frac{1}{2}\int (1 - x^2)^{-3}(-2x)\,dx = -\frac{1}{2}\frac{(1 - x^2)^{-2}}{-2} + C = \frac{1}{4(1 - x^2)^2} + C$

Check: $\frac{d}{dx}\left[\frac{1}{4(1 - x^2)^2} + C\right] = \frac{1}{4}(-2)(1 - x^2)^{-3}(-2x) = \frac{x}{(1 - x^2)^3}$

20. $\int \frac{x^3}{(1 + x^4)^2}\,dx = \frac{1}{4}\int (1 + x^4)^{-2}(4x^3)\,dx = -\frac{1}{4}(1 + x^4)^{-1} + C = \frac{-1}{4(1 + x^4)} + C$

Check: $\frac{d}{dx}\left[\frac{-1}{4(1 + x^4)} + C\right] = \frac{1}{4}(1 + x^4)^{-2}(4x^3) = \frac{x^3}{(1 + x^4)^2}$

21. $\int \frac{x^2}{(1 + x^3)^2}\,dx = \frac{1}{3}\int (1 + x^3)^{-2}(3x^2)\,dx = \frac{1}{3}\left[\frac{(1 + x^3)^{-1}}{-1}\right] + C = -\frac{1}{3(1 + x^3)} + C$

Check: $\frac{d}{dx}\left[-\frac{1}{3(1 + x^3)} + C\right] = -\frac{1}{3}(-1)(1 + x^3)^{-2}(3x^2) = \frac{x^2}{(1 + x^3)^2}$

22. $\int \frac{x^2}{(16 - x^3)^2}\,dx = -\frac{1}{3}\int (16 - x^3)^{-2}(-3x^2)\,dx = -\frac{1}{3}\left[\frac{(16 - x^3)^{-1}}{-1}\right] + C = \frac{1}{3(16 - x^3)} + C$

Check: $\frac{d}{dx}\left[\frac{1}{3(16 - x^3)} + C\right] = \frac{1}{3}(-1)(16 - x^3)^{-2}(3x^2) = \frac{x^2}{(16 - x^3)^2}$

23. $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\,dx = -\frac{1}{2}\frac{(1-x^2)^{1/2}}{1/2} + C = -\sqrt{1-x^2} + C$

Check: $\displaystyle\frac{d}{dx}\left[-(1-x^2)^{1/2} + C\right] = -\frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{1-x^2}}$

24. $\displaystyle\int \frac{x^3}{\sqrt{1+x^4}}\,dx = \frac{1}{4}\int (1+x^4)^{-1/2}(4x^3)\,dx = \frac{1}{4}\frac{(1+x^4)^{1/2}}{1/2} + C = \frac{\sqrt{1+x^4}}{2} + C$

Check: $\displaystyle\frac{d}{dx}\left[\frac{\sqrt{1+x^4}}{2} + C\right] = \frac{1}{2}\cdot\frac{1}{2}(1+x^4)^{-1/2}(4x^3) = \frac{x^3}{\sqrt{1+x^4}}$

25. $\displaystyle\int \left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt = -\int \left(1+\frac{1}{t}\right)^3\left(-\frac{1}{t^2}\right)dt = -\frac{[1+(1/t)]^4}{4} + C$

Check: $\displaystyle\frac{d}{dt}\left[-\frac{[1+(1/t)]^4}{4} + C\right] = -\frac{1}{4}(4)\left(1+\frac{1}{t}\right)^3\left(-\frac{1}{t^2}\right) = \frac{1}{t^2}\left(1+\frac{1}{t}\right)^3$

26. $\displaystyle\int \left[x^2 + \frac{1}{(3x)^2}\right]dx = \int \left(x^2 + \frac{1}{9}x^{-2}\right)dx = \frac{x^3}{3} + \frac{1}{9}\left(\frac{x^{-1}}{-1}\right) + C = \frac{x^3}{3} - \frac{1}{9x} + C = \frac{3x^4-1}{9x} + C$

Check: $\displaystyle\frac{d}{dx}\left[\frac{1}{3}x^3 - \frac{1}{9}x^{-1} + C\right] = x^2 + \frac{1}{9}x^{-2} = x^2 + \frac{1}{(3x)^2}$

27. $\displaystyle\int \frac{1}{\sqrt{2x}}\,dx = \frac{1}{2}\int (2x)^{-1/2}\,2\,dx = \frac{1}{2}\left[\frac{(2x)^{1/2}}{1/2}\right] + C = \sqrt{2x} + C$

Alternate Solution: $\displaystyle\int \frac{1}{\sqrt{2x}}\,dx = \frac{1}{\sqrt{2}}\int x^{-1/2}\,dx = \frac{1}{\sqrt{2}}\frac{x^{1/2}}{(1/2)} + C = \sqrt{2x} + C$

Check: $\displaystyle\frac{d}{dx}\left[\sqrt{2x} + C\right] = \frac{1}{2}(2x)^{-1/2}(2) = \frac{1}{\sqrt{2x}}$

28. $\displaystyle\int \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2}\int x^{-1/2}\,dx = \frac{1}{2}\left(\frac{x^{1/2}}{1/2}\right) + C = \sqrt{x} + C$

Check: $\displaystyle\frac{d}{dx}\left[\sqrt{x} + C\right] = \frac{1}{2\sqrt{x}}$

29. $\displaystyle\int \frac{x^2+3x+7}{\sqrt{x}}\,dx = \int (x^{3/2} + 3x^{1/2} + 7x^{-1/2})\,dx = \frac{2}{5}x^{5/2} + 2x^{3/2} + 14x^{1/2} + C = \frac{2}{5}\sqrt{x}(x^2+5x+35) + C$

Check: $\displaystyle\frac{d}{dx}\left[\frac{2}{5}x^{5/2} + 2x^{3/2} + 14x^{1/2} + C\right] = \frac{x^2+3x+7}{\sqrt{x}}$

30. $\displaystyle\int \frac{t+2t^2}{\sqrt{t}}\,dt = \int (t^{1/2} + 2t^{3/2})\,dt = \frac{2}{3}t^{3/2} + \frac{4}{5}t^{5/2} + C = \frac{2}{15}t^{3/2}(5+6t) + C$

Check: $\displaystyle\frac{d}{dt}\left[\frac{2}{3}t^{3/2} + \frac{4}{5}t^{5/2} + C\right] = t^{1/2} + 2t^{3/2} = \frac{t+2t^2}{\sqrt{t}}$

31. $\displaystyle\int t^2\left(t - \frac{2}{t}\right)dt = \int (t^3 - 2t)\,dt = \frac{1}{4}t^4 - t^2 + C$

Check: $\displaystyle\frac{d}{dt}\left[\frac{1}{4}t^4 - t^2 + C\right] = t^3 - 2t = t^2\left(t - \frac{2}{t}\right)$

32. $\int\left(\frac{t^3}{3} + \frac{1}{4t^2}\right) dt = \int\left(\frac{1}{3}t^3 + \frac{1}{4}t^{-2}\right) dt = \frac{1}{3}\left(\frac{t^4}{4}\right) + \frac{1}{4}\left(\frac{t^{-1}}{-1}\right) + C = \frac{1}{12}t^4 - \frac{1}{4t} + C$

Check: $\frac{d}{dt}\left[\frac{1}{12}t^4 - \frac{1}{4t} + C\right] = \frac{1}{3}t^3 + \frac{1}{4t^2}$

33. $\int(9 - y)\sqrt{y}\, dy = \int(9y^{1/2} - y^{3/2})\, dy = 9\left(\frac{2}{3}y^{3/2}\right) - \frac{2}{5}y^{5/2} + C = \frac{2}{5}y^{3/2}(15 - y) + C$

Check: $\frac{d}{dy}\left[\frac{2}{5}y^{3/2}(15 - y) + C\right] = \frac{d}{dy}\left[6y^{3/2} - \frac{2}{5}y^{5/2} + C\right] = 9y^{1/2} - y^{3/2} = (9 - y)\sqrt{y}$

34. $\int 2\pi y(8 - y^{3/2})\, dy = 2\pi\int(8y - y^{5/2})\, dy = 2\pi\left(4y^2 - \frac{2}{7}y^{7/2}\right) + C = \frac{4\pi y^2}{7}(14 - y^{3/2}) + C$

Check: $\frac{d}{dy}\left[\frac{4\pi y^2}{7}(14 - y^{3/2}) + C\right] = \frac{d}{dy}\left[2\pi\left(4y^2 - \frac{2}{7}y^{7/2}\right) + C\right] = 16\pi y - 2\pi y^{5/2} = (2\pi y)(8 - y^{3/2})$

35. $y = \int\left[4x + \frac{4x}{\sqrt{16 - x^2}}\right] dx$

$= 4\int x\, dx - 2\int(16 - x^2)^{-1/2}(-2x)\, dx$

$= 4\left(\frac{x^2}{2}\right) - 2\left[\frac{(16 - x^2)^{1/2}}{1/2}\right] + C$

$= 2x^2 - 4\sqrt{16 - x^2} + C$

36. $y = \int\frac{10x^2}{\sqrt{1 + x^3}}\, dx$

$= \frac{10}{3}\int(1 + x^3)^{-1/2}(3x^2)\, dx$

$= \frac{10}{3}\left[\frac{(1 + x^3)^{1/2}}{1/2}\right] + C$

$= \frac{20}{3}\sqrt{1 + x^3} + C$

37. $y = \int\frac{x + 1}{(x^2 + 2x - 3)^2}\, dx$

$= \frac{1}{2}\int(x^2 + 2x - 3)^{-2}(2x + 2)\, dx$

$= \frac{1}{2}\left[\frac{(x^2 + 2x - 3)^{-1}}{-1}\right] + C$

$= -\frac{1}{2(x^2 + 2x - 3)} + C$

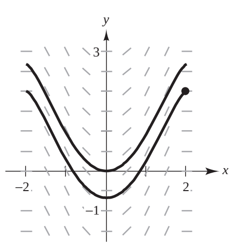
38. $y = \int\frac{x - 4}{\sqrt{x^2 - 8x + 1}}\, dx$

$= \frac{1}{2}\int(x^2 - 8x + 1)^{-1/2}(2x - 8)\, dx$

$= \frac{1}{2}\left[\frac{(x^2 - 8x + 1)^{1/2}}{1/2}\right] + C$

$= \sqrt{x^2 - 8x + 1} + C$

39. (a)



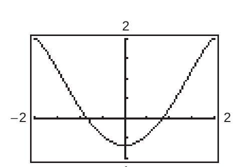
(b) $\frac{dy}{dx} = x\sqrt{4 - x^2}$, $(2, 2)$

$y = \int x\sqrt{4 - x^2}\, dx = -\frac{1}{2}\int(4 - x^2)^{1/2}(-2x\, dx)$

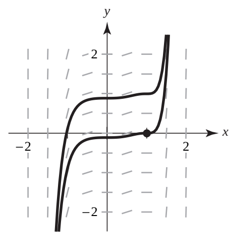
$= -\frac{1}{2}\cdot\frac{2}{3}(4 - x^2)^{3/2} + C = -\frac{1}{3}(4 - x^2)^{3/2} + C$

$(2, 2)$: $2 = -\frac{1}{3}(4 - 2^2)^{3/2} + C \implies C = 2$

$y = -\frac{1}{3}(4 - x^2)^{3/2} + 2$



40. (a)



(b) $\dfrac{dy}{dx} = x^2(x^3 - 1)^2,\ (1, 0)$

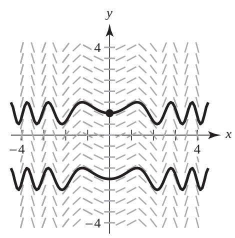
$$y = \int x^2(x^3 - 1)^2 dx = \frac{1}{3}\int (x^3 - 1)^2(3x^2\,dx)\ (u = x^3 - 1)$$

$$= \frac{1}{3}\frac{(x^3 - 1)^3}{3} + C = \frac{1}{9}(x^3 - 1)^3 + C$$

$$0 = C$$

$$y = \frac{1}{9}(x^3 - 1)^3$$

41. (a)



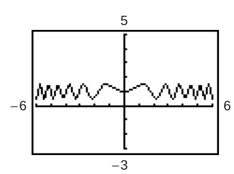
(b) $\dfrac{dy}{dx} = x\cos x^2,\ (0, 1)$

$$y = \int x\cos x^2\,dx = \frac{1}{2}\int \cos(x^2)2x\,dx$$

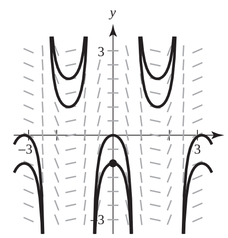
$$= \frac{1}{2}\sin(x^2) + C$$

$$(0, 1):\ 1 = \frac{1}{2}\sin(0) + C \implies C = 1$$

$$y = \frac{1}{2}\sin(x^2) + 1$$



42. (a)



(b) $\dfrac{dy}{dx} = -2\sec(2x)\tan(2x),\ (0, -1)$

$$y = \int -2\sec(2x)\tan(2x)\,dx\ (u = 2x)$$

$$= -\sec(2x) + C$$

$$-1 = -\sec(0) + C \implies C = 0$$

$$y = -\sec(2x)$$

43. $\displaystyle\int \pi\sin\pi x\,dx = -\cos\pi x + C$

44. $\displaystyle\int 4x^3\sin x^4\,dx = \int \sin x^4(4x^3)\,dx = -\cos x^4 + C$

45. $\displaystyle\int \sin 2x\,dx = \frac{1}{2}\int(\sin 2x)(2x)\,dx = -\frac{1}{2}\cos 2x + C$

46. $\displaystyle\int \cos 6x\,dx = \frac{1}{6}\int(\cos 6x)(6)\,dx = \frac{1}{6}\sin 6x + C$

47. $\displaystyle\int \frac{1}{\theta^2}\cos\frac{1}{\theta}\,d\theta = -\int \cos\frac{1}{\theta}\left(-\frac{1}{\theta^2}\right)d\theta = -\sin\frac{1}{\theta} + C$

48. $\displaystyle\int x\sin x^2\,dx = \frac{1}{2}\int(\sin x^2)(2x)\,dx = -\frac{1}{2}\cos x^2 + C$

49. $\displaystyle\int \sin 2x\cos 2x\,dx = \frac{1}{2}\int(\sin 2x)(2\cos 2x)\,dx = \frac{1}{2}\frac{(\sin 2x)^2}{2} + C = \frac{1}{4}\sin^2 2x + C$ OR

$\displaystyle\int \sin 2x\cos 2x\,dx = -\frac{1}{2}\int(\cos 2x)(-2\sin 2x)\,dx = -\frac{1}{2}\frac{(\cos 2x)^2}{2} + C_1 = -\frac{1}{4}\cos^2 2x + C_1$ OR

$\displaystyle\int \sin 2x\cos 2x\,dx = \frac{1}{2}\int 2\sin 2x\cos 2x\,dx = \frac{1}{2}\int \sin 4x\,dx = -\frac{1}{8}\cos 4x + C_2$

50. $\int \sec(1-x)\tan(1-x)\,dx = -\int [\sec(1-x)\tan(1-x)](-1)\,dx = -\sec(1-x) + C$

51. $\int \tan^4 x \sec^2 x\,dx = \dfrac{\tan^5 x}{5} + C = \dfrac{1}{5}\tan^5 x + C$

52. $\int \sqrt{\tan x}\sec^2 x\,dx = \dfrac{(\tan x)^{3/2}}{3/2} + C = \dfrac{2}{3}(\tan x)^{3/2} + C$

53. $\int \dfrac{\csc^2 x}{\cot^3 x}\,dx = -\int (\cot x)^{-3}(-\csc^2 x)\,dx$

$$= -\frac{(\cot x)^{-2}}{-2} + C = \frac{1}{2\cot^2 x} + C = \frac{1}{2}\tan^2 x + C = \frac{1}{2}(\sec^2 x - 1) + C = \frac{1}{2}\sec^2 x + C_1$$

54. $\int \dfrac{\sin x}{\cos^3 x}\,dx = -\int (\cos x)^{-3}(-\sin x)\,dx = -\dfrac{(\cos x)^{-2}}{-2} + C = \dfrac{1}{2\cos^2 x} + C = \dfrac{1}{2}\sec^2 x + C$

55. $\int \cot^2 x\,dx = \int (\csc^2 x - 1)\,dx = -\cot x - x + C$

56. $\int \csc^2\left(\dfrac{x}{2}\right)dx = 2\int \csc^2\left(\dfrac{x}{2}\right)\left(\dfrac{1}{2}\right)dx = -2\cot\left(\dfrac{x}{2}\right) + C$

57. $f(x) = \int \cos\dfrac{x}{2}\,dx = 2\sin\dfrac{x}{2} + C$

Since $f(0) = 3 = 2\sin 0 + C$, $C = 3$. Thus,

$$f(x) = 2\sin\frac{x}{2} + 3.$$

58. $f(x) = \int \pi\sec \pi x \tan \pi x\,dx = \sec \pi x + C$

Since $f(1/3) = 1 = \sec(\pi/3) + C$, $C = -1$. Thus

$$f(x) = \sec \pi x - 1.$$

59. $f'(x) = \sin 4x, \left(\dfrac{\pi}{4}, \dfrac{-3}{4}\right)$

$$f(x) = \frac{-1}{4}\cos 4x + C$$

$$f\left(\frac{\pi}{4}\right) = \frac{-1}{4}\cos\left(4\left(\frac{\pi}{4}\right)\right) + C = \frac{-3}{4}$$

$$-\frac{1}{4}(-1) + C = \frac{-3}{4}$$

$$C = -1$$

$$f(x) = -\frac{1}{4}\cos 4x - 1$$

60. $f'(x) = \sec^2(2x), \left(\dfrac{\pi}{2}, 2\right)$

$$f(x) = \frac{1}{2}\tan(2x) + C$$

$$f\left(\frac{\pi}{2}\right) = \frac{1}{2}\tan\left(2\left(\frac{\pi}{2}\right)\right) + C = 2$$

$$\frac{1}{2}(0) + C = 2$$

$$C = 2$$

$$f(x) = \frac{1}{2}\tan(2x) + 2$$

61. $f'(x) = 2x(4x^2 - 10)^2, (2, 10)$

$$f(x) = \frac{(4x^2-10)^3}{12} + C = \frac{2(2x^2-5)^3}{3} + C$$

$$f(2) = \frac{2(8-5)^3}{3} + C = 18 + C = 10 \implies C = -8$$

$$f(x) = \frac{2}{3}(2x^2-5)^3 - 8$$

62. $f'(x) = -2x\sqrt{8 - x^2}, (2, 7)$

$$f(x) = \frac{2(8-x^2)^{3/2}}{3} + C$$

$$f(2) = \frac{2(4)^{3/2}}{3} + C = \frac{16}{3} + C = 7 \implies C = \frac{5}{3}$$

$$f(x) = \frac{2(8-x^2)^{3/2}}{3} + \frac{5}{3}$$

63. $u = x + 2,\ x = u - 2,\ dx = du$

$$\begin{aligned}\int x\sqrt{x+2}\,dx &= \int (u-2)\sqrt{u}\,du\\ &= \int \left(u^{3/2} - 2u^{1/2}\right) du\\ &= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C\\ &= \frac{2u^{3/2}}{15}(3u - 10) + C\\ &= \frac{2}{15}(x+2)^{3/2}[3(x+2) - 10] + C\\ &= \frac{2}{15}(x+2)^{3/2}(3x - 4) + C\end{aligned}$$

64. $u = 2x + 1,\ x = \frac{1}{2}(u - 1),\ dx = \frac{1}{2}\,du$

$$\begin{aligned}\int x\sqrt{2x+1}\,dx &= \int \frac{1}{2}(u-1)\sqrt{u}\,\frac{1}{2}\,du\\ &= \frac{1}{4}\int \left(u^{3/2} - u^{1/2}\right) du\\ &= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C\\ &= \frac{u^{3/2}}{30}(3u - 5) + C\\ &= \frac{1}{30}(2x+1)^{3/2}[3(2x+1) - 5] + C\\ &= \frac{1}{30}(2x+1)^{3/2}(6x - 2) + C\\ &= \frac{1}{15}(2x+1)^{3/2}(3x - 1) + C\end{aligned}$$

65. $u = 1 - x,\ x = 1 - u,\ dx = -du$

$$\begin{aligned}\int x^2\sqrt{1-x}\,dx &= -\int (1-u)^2\sqrt{u}\,du\\ &= -\int \left(u^{1/2} - 2u^{3/2} + u^{5/2}\right) du\\ &= -\left(\frac{2}{3}u^{3/2} - \frac{4}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right) + C\\ &= -\frac{2u^{3/2}}{105}(35 - 42u + 15u^2) + C\\ &= -\frac{2}{105}(1-x)^{3/2}[35 - 42(1-x) + 15(1-x)^2] + C\\ &= -\frac{2}{105}(1-x)^{3/2}(15x^2 + 12x + 8) + C\end{aligned}$$

66. $u = 2 - x,\ x = 2 - u,\ dx = -du$

$$\begin{aligned}\int (x+1)\sqrt{2-x}\,dx &= -\int (3-u)\sqrt{u}\,du\\ &= -\int \left(3u^{1/2} - u^{3/2}\right) du\\ &= -\left(2u^{3/2} - \frac{2}{5}u^{5/2}\right) + C\\ &= -\frac{2u^{3/2}}{5}(5 - u) + C\\ &= -\frac{2}{5}(2-x)^{3/2}[5 - (2-x)] + C\\ &= -\frac{2}{5}(2-x)^{3/2}(x + 3) + C\end{aligned}$$

67. $u = 2x - 1, x = \frac{1}{2}(u + 1), dx = \frac{1}{2}\,du$

$$\begin{aligned}\int \frac{x^2 - 1}{\sqrt{2x - 1}}\,dx &= \int \frac{[(1/2)(u + 1)]^2 - 1}{\sqrt{u}}\frac{1}{2}\,du \\ &= \frac{1}{8}\int u^{-1/2}[(u^2 + 2u + 1) - 4]\,du \\ &= \frac{1}{8}\int (u^{3/2} + 2u^{1/2} - 3u^{-1/2})\,du \\ &= \frac{1}{8}\left(\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2} - 6u^{1/2}\right) + C \\ &= \frac{u^{1/2}}{60}(3u^2 + 10u - 45) + C \\ &= \frac{\sqrt{2x - 1}}{60}\left[3(2x - 1)^2 + 10(2x - 1) - 45\right] + C \\ &= \frac{1}{60}\sqrt{2x - 1}(12x^2 + 8x - 52) + C \\ &= \frac{1}{15}\sqrt{2x - 1}(3x^2 + 2x - 13) + C\end{aligned}$$

68. Let $u = x + 4, x = u - 4, du = dx.$

$$\begin{aligned}\int \frac{2x + 1}{\sqrt{x + 4}}\,dx &= \int \frac{2(u - 4) + 1}{\sqrt{u}}\,du \\ &= \int (2u^{1/2} - 7u^{-1/2})\,du \\ &= \frac{4}{3}u^{3/2} - 14u^{1/2} + C \\ &= \frac{2}{3}u^{1/2}(2u - 21) + C \\ &= \frac{2}{3}\sqrt{x + 4}\left[2(x + 4) - 21\right] + C \\ &= \frac{2}{3}\sqrt{x + 4}(2x - 13) + C\end{aligned}$$

69. $u = x + 1, x = u - 1, dx = du$

$$\begin{aligned}\int \frac{-x}{(x + 1) - \sqrt{x + 1}}\,dx &= \int \frac{-(u - 1)}{u - \sqrt{u}}\,du \\ &= -\int \frac{\left(\sqrt{u} + 1\right)\left(\sqrt{u} - 1\right)}{\sqrt{u}\left(\sqrt{u} - 1\right)}\,du \\ &= -\int (1 + u^{-1/2})\,du \\ &= -(u + 2u^{1/2}) + C \\ &= -u - 2\sqrt{u} + C \\ &= -(x + 1) - 2\sqrt{x + 1} + C \\ &= -x - 2\sqrt{x + 1} - 1 + C \\ &= -\left(x + 2\sqrt{x + 1}\right) + C_1\end{aligned}$$

where $C_1 = -1 + C$.

70. $u = t - 4, t = u + 4, dt = du$

$$\begin{aligned}\int t\sqrt[3]{t - 4}\,dt &= \int (u + 4)u^{1/3}\,du \\ &= \int (u^{4/3} + 4u^{1/3})\,du \\ &= \frac{3}{7}u^{7/3} + 3u^{4/3} + C \\ &= \frac{3u^{4/3}}{7}(u + 7) + C \\ &= \frac{3}{7}(t - 4)^{4/3}\left[(t - 4) + 7\right] + C \\ &= \frac{3}{7}(t - 4)^{4/3}(t + 3) + C\end{aligned}$$

71. Let $u = x^2 + 1$, $du = 2x\,dx$.

$$\int_{-1}^{1} x(x^2 + 1)^3\,dx = \frac{1}{2}\int_{-1}^{1} (x^2 + 1)^3(2x)\,dx = \left[\frac{1}{8}(x^2 + 1)^4\right]_{-1}^{1} = 0$$

72. Let $u = x^3 + 8$, $du = 3x^2\,dx$.

$$\int_{-2}^{4} x^2(x^3 + 8)^2\,dx = \frac{1}{3}\int_{-2}^{4} (x^3 + 8)^2(3x^2)\,dx = \left[\frac{1}{3}\frac{(x^3 + 8)^3}{3}\right]_{-2}^{4}$$

$$= \frac{1}{9}\left[(64 + 8)^3 - (-8 + 8)^3\right] = 41{,}472$$

73. Let $u = x^3 + 1$, $du = 3x^2\,dx$.

$$\int_{1}^{2} 2x^2\sqrt{x^3 + 1}\,dx = 2 \cdot \frac{1}{3}\int_{1}^{2} (x^3 + 1)^{1/2}(3x^2)\,dx$$

$$= \left[\frac{2}{3}\frac{(x^3 + 1)^{3/2}}{3/2}\right]_{1}^{2}$$

$$= \frac{4}{9}\left[(x^3 + 1)^{3/2}\right]_{1}^{2}$$

$$= \frac{4}{9}\left[27 - 2\sqrt{2}\right] = 12 - \frac{8}{9}\sqrt{2}$$

74. Let $u = 1 - x^2$, $du = -2x\,dx$.

$$\int_{0}^{1} x\sqrt{1 - x^2}\,dx = -\frac{1}{2}\int_{0}^{1} (1 - x^2)^{1/2}(-2x)\,dx = \left[-\frac{1}{3}(1 - x^2)^{3/2}\right]_{0}^{1} = 0 + \frac{1}{3} = \frac{1}{3}$$

75. Let $u = 2x + 1$, $du = 2\,dx$.

$$\int_{0}^{4} \frac{1}{\sqrt{2x + 1}}\,dx = \frac{1}{2}\int_{0}^{4} (2x + 1)^{-1/2}(2)\,dx = \left[\sqrt{2x + 1}\right]_{0}^{4} = \sqrt{9} - \sqrt{1} = 2$$

76. Let $u = 1 + 2x^2$, $du = 4x\,dx$.

$$\int_{0}^{2} \frac{x}{\sqrt{1 + 2x^2}}\,dx = \frac{1}{4}\int_{0}^{2} (1 + 2x^2)^{-1/2}(4x)\,dx = \left[\frac{1}{2}\sqrt{1 + 2x^2}\right]_{0}^{2} = \frac{3}{2} - \frac{1}{2} = 1$$

77. Let $u = 1 + \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$.

$$\int_{1}^{9} \frac{1}{\sqrt{x}(1 + \sqrt{x})^2}\,dx = 2\int_{1}^{9} (1 + \sqrt{x})^{-2}\left(\frac{1}{2\sqrt{x}}\right)dx = \left[-\frac{2}{1 + \sqrt{x}}\right]_{1}^{9} = -\frac{1}{2} + 1 = \frac{1}{2}$$

78. Let $u = 4 + x^2$, $du = 2x\,dx$.

$$\int_{0}^{2} x\sqrt[3]{4 + x^2}\,dx = \frac{1}{2}\int_{0}^{2} (4 + x^2)^{1/3}(2x)\,dx = \left[\frac{3}{8}(4 + x^2)^{4/3}\right]_{0}^{2} = \frac{3}{8}(8^{4/3} - 4^{4/3}) = 6 - \frac{3}{2}\sqrt[3]{4} \approx 3.619$$

79. $u = 2 - x$, $x = 2 - u$, $dx = -du$

When $x = 1$, $u = 1$. When $x = 2$, $u = 0$.

$$\int_{1}^{2} (x - 1)\sqrt{2 - x}\,dx = \int_{1}^{0} -\left[(2 - u) - 1\right]\sqrt{u}\,du = \int_{1}^{0} (u^{3/2} - u^{1/2})\,du = \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_{1}^{0} = -\left[\frac{2}{5} - \frac{2}{3}\right] = \frac{4}{15}$$

80. Let $u = 2x - 1$, $du = 2\,dx$, $x = \frac{1}{2}(u + 1)$.

When $x = 1$, $u = 1$. When $x = 5$, $u = 9$.

$$\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx = \int_1^9 \frac{1/2(u+1)}{\sqrt{u}}\frac{1}{2}\,du = \frac{1}{4}\int_1^9 (u^{1/2} + u^{-1/2})\,du$$
$$= \frac{1}{4}\left[\frac{2}{3}u^{3/2} + 2u^{1/2}\right]_1^9$$
$$= \frac{1}{4}\left[\left(\frac{2}{3}(27) + 2(3)\right) - \left(\frac{2}{3} + 2\right)\right]$$
$$= \frac{16}{3}$$

81. $\displaystyle\int_0^{\pi/2} \cos\left(\frac{2}{3}x\right)dx = \left[\frac{3}{2}\sin\left(\frac{2}{3}x\right)\right]_0^{\pi/2} = \frac{3}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{4}$

82. $\displaystyle\int_{\pi/3}^{\pi/2} (x + \cos x)\,dx = \left[\frac{x^2}{2} + \sin x\right]_{\pi/3}^{\pi/2} = \left(\frac{\pi^2}{8} + 1\right) - \left(\frac{\pi^2}{18} + \frac{\sqrt{3}}{2}\right) = \frac{5\pi^2}{72} + \frac{2 - \sqrt{3}}{2}$

83. $\dfrac{dy}{dx} = 18x^2(2x^3 + 1)^2$, $(0, 4)$

$$y = 3\int (2x^3 + 1)^2\,(6x^2)\,dx \quad (u = 2x^3 + 1)$$
$$y = 3\frac{(2x^3 + 1)^3}{3} + C = (2x^3 + 1)^3 + C$$
$$4 = 1^3 + C \implies C = 3$$
$$y = (2x^3 + 1)^3 + 3$$

84. $\dfrac{dy}{dx} = \dfrac{-48}{(3x + 5)^3}$, $(-1, 3)$

$$y = -48\int (3x + 5)^{-3}\,dx$$
$$= (-48)\frac{1}{3}\int (3x + 5)^{-3}\,3\,dx$$
$$= \frac{-16(3x + 5)^{-2}}{-2} + C$$
$$= \frac{8}{(3x + 5)^2} + C$$
$$3 = \frac{8}{(3(-1) + 5)^2} + C = \frac{8}{4} + C \implies C = 1$$
$$y = \frac{8}{(3x + 5)^2} + 1$$

85. $\dfrac{dy}{dx} = \dfrac{2x}{\sqrt{2x^2 - 1}}$, $(5, 4)$

$$y = \frac{1}{2}\int (2x^2 - 1)^{-1/2}\,(4x\,dx) \quad (u = 2x^2 - 1)$$
$$y = \frac{1}{2}\frac{(2x^2 - 1)^{1/2}}{1/2} + C = \sqrt{2x^2 - 1} + C$$
$$4 = \sqrt{49} + C = 7 + C \implies C = -3$$
$$y = \sqrt{2x^2 - 1} - 3$$

86. $\dfrac{dy}{dx} = 4x + \dfrac{9x^2}{(3x^3 + 1)^{3/2}}$, $(0, 2)$

$$y = \int \left(4x + (3x^3 + 1)^{-3/2}\,9x^2\right)dx$$
$$= 2x^2 + \frac{(3x^3 + 1)^{-1/2}}{(-1/2)} + C$$
$$= 2x^2 - \frac{2}{\sqrt{3x^3 + 1}} + C$$
$$2 = 0 - \frac{2}{1} + C \implies C = 4$$
$$y = 2x^2 - \frac{2}{\sqrt{3x^3 + 1}} + 4$$

87. $u = x + 1, x = u - 1, dx = du$

When $x = 0, u = 1$. When $x = 7, u = 8$.

$$\text{Area} = \int_0^7 x\sqrt[3]{x+1}\,dx = \int_1^8 (u-1)\sqrt[3]{u}\,du$$

$$= \int_1^8 (u^{4/3} - u^{1/3})\,du = \left[\frac{3}{7}u^{7/3} - \frac{3}{4}u^{4/3}\right]_1^8 = \left(\frac{384}{7} - 12\right) - \left(\frac{3}{7} - \frac{3}{4}\right) = \frac{1209}{28}$$

88. $u = x + 2, x = u - 2, dx = du$

When $x = -2, u = 0$. When $x = 6, u = 8$.

$$\text{Area} = \int_{-2}^6 x^2\sqrt[3]{x+2}\,dx = \int_0^8 (u-2)^2\sqrt[3]{u}\,du = \int_0^8 (u^{7/3} - 4u^{4/3} + 4u^{1/3})\,du = \left[\frac{3}{10}u^{10/3} - \frac{12}{7}u^{7/3} + 3u^{4/3}\right]_0^8 = \frac{4752}{35}$$

89. $A = \int_0^\pi (2\sin x + \sin 2x)\,dx = -\left[2\cos x + \frac{1}{2}\cos 2x\right]_0^\pi = 4$

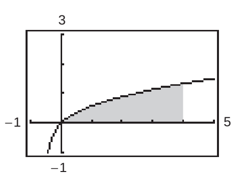
90. $A = \int_0^\pi (\sin x + \cos 2x)\,dx = \left[-\cos x + \frac{1}{2}\sin 2x\right]_0^\pi = 2$

91. $\text{Area} = \int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right)dx = 2\int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)dx = \left[2\tan\left(\frac{x}{2}\right)\right]_{\pi/2}^{2\pi/3} = 2\left(\sqrt{3} - 1\right)$

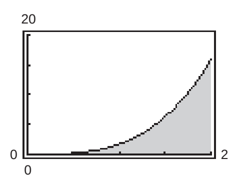
92. Let $u = 2x, du = 2\,dx$.

$$\text{Area} = \int_{\pi/12}^{\pi/4} \csc 2x \cot 2x\,dx = \frac{1}{2}\int_{\pi/12}^{\pi/4} \csc 2x \cot 2x(2)\,dx = \left[-\frac{1}{2}\csc 2x\right]_{\pi/12}^{\pi/4} = \frac{1}{2}$$

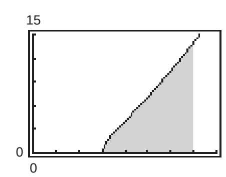
93. $\int_0^4 \frac{x}{\sqrt{2x+1}}\,dx \approx 3.333 = \frac{10}{3}$



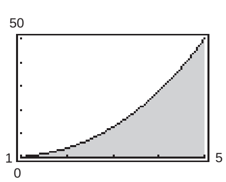
94. $\int_0^2 x^3\sqrt{x+2}\,dx \approx 7.581$



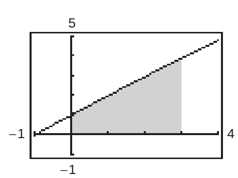
95. $\int_3^7 x\sqrt{x-3}\,dx \approx 28.8 = \frac{144}{5}$



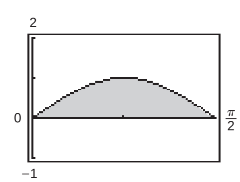
96. $\int_1^5 x^2\sqrt{x-1}\,dx \approx 67.505$



97. $\int_0^3 \left(\theta + \cos\frac{\theta}{6}\right)d\theta \approx 7.377$



98. $\int_0^{\pi/2} \sin 2x\,dx \approx 1.0$



99. $\int (2x-1)^2\,dx = \frac{1}{2}\int (2x-1)^2\,2\,dx = \frac{1}{6}(2x-1)^3 + C_1 = \frac{4}{3}x^3 - 2x^2 + x - \frac{1}{6} + C_1$

$\int (2x-1)^2\,dx = \int (4x^2 - 4x + 1)\,dx = \frac{4}{3}x^3 - 2x^2 + x + C_2$

They differ by a constant: $C_2 = C_1 - \frac{1}{6}$.

100. $\int \sin x \cos x\,dx = \int (\sin x)^1(\cos x\,dx) = \dfrac{\sin^2 x}{2} + C_1$

$\int \sin x \cos x\,dx = -\int (\cos x)^1(-\sin x\,dx) = -\dfrac{\cos^2 x}{2} + C_2$

$-\dfrac{\cos^2 x}{2} + C_2 = -\dfrac{(1 - \sin^2 x)}{2} + C_2 = \dfrac{\sin^2 x}{2} - \dfrac{1}{2} + C_2$

They differ by a constant: $C_2 = C_1 + \dfrac{1}{2}$.

101. $f(x) = x^2(x^2 + 1)$ is even.

$\int_{-2}^{2} x^2(x^2 + 1)\,dx = 2\int_0^2 (x^4 + x^2)\,dx = 2\left[\dfrac{x^5}{5} + \dfrac{x^3}{3}\right]_0^2$

$= 2\left[\dfrac{32}{5} + \dfrac{8}{3}\right] = \dfrac{272}{15}$

102. $f(x) = x(x^2 + 1)^3$ is odd.

$\int_{-2}^{2} x(x^2 + 1)^3\,dx = 0$

103. $f(x) = \sin^2 x \cos x$ is even.

$\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\,dx = \int_0^{\pi/2} \sin^2 x(\cos x)\,dx$

$= 2\left[\dfrac{\sin^3 x}{3}\right]_0^{\pi/2}$

$= \dfrac{2}{3}$

104. $f(x) = \sin x \cos x$ is odd.

$\int_{-\pi/2}^{\pi/2} \sin x \cos x\,dx = 0$

105. $\int_0^2 x^2\,dx = \left[\dfrac{x^3}{3}\right]_0^2 = \dfrac{8}{3}$; the function x^2 is an even function.

(a) $\int_{-2}^{0} x^2\,dx = \int_0^2 x^2\,dx = \dfrac{8}{3}$

(b) $\int_{-2}^{2} x^2\,dx = 2\int_0^2 x^2\,dx = \dfrac{16}{3}$

(c) $\int_0^2 (-x^2)\,dx = -\int_0^2 x^2\,dx = -\dfrac{8}{3}$

(d) $\int_{-2}^{0} 3x^2\,dx = 3\int_0^2 x^2\,dx = 8$

106. (a) $\int_{-\pi/4}^{\pi/4} \sin x\,dx = 0$ since $\sin x$ is symmetric to the origin.

(b) $\int_{-\pi/4}^{\pi/4} \cos x\,dx = 2\int_0^{\pi/4} \cos x\,dx = \Big[2\sin x\Big]_0^{\pi/4} = \sqrt{2}$ since $\cos x$ is symmetric to the y-axis.

(c) $\int_{-\pi/2}^{\pi/2} \cos x\,dx = 2\int_0^{\pi/2} \cos x\,dx = \Big[2\sin x\Big]_0^{\pi/2} = 2$

(d) $\int_{-\pi/2}^{\pi/2} \sin x \cos x\,dx = 0$ since $\sin(-x)\cos(-x) = -\sin x \cos x$ and hence, is symmetric to the origin.

107. $\int_{-4}^{4} (x^3 + 6x^2 - 2x - 3)\,dx = \int_{-4}^{4} (x^3 - 2x)\,dx + \int_{-4}^{4} (6x^2 - 3)\,dx = 0 + 2\int_0^4 (6x^2 - 3)\,dx = 2\Big[2x^3 - 3x\Big]_0^4 = 232$

108. $\int_{-\pi}^{\pi} (\sin 3x + \cos 3x)\,dx = \int_{-\pi}^{\pi} \sin 3x\,dx + \int_{-\pi}^{\pi} \cos 3x\,dx = 0 + 2\int_0^{\pi} \cos 3x\,dx = \left[\dfrac{2}{3}\sin 3x\right]_0^{\pi} = 0$

109. If $u = 5 - x^2$, then $du = -2x\,dx$ and $\int x(5 - x^2)^3\,dx = -\frac{1}{2}\int (5 - x^2)^3(-2x)\,dx = -\frac{1}{2}\int u^3\,du$.

110. $f(x) = x(x^2 + 1)^2$ is odd. Hence, $\int_{-2}^{2} x(x^2 + 1)^2\,dx = 0$.

111. $\frac{dQ}{dt} = k(100 - t)^2$

$$Q(t) = \int k(100 - t)^2\,dt = -\frac{k}{3}(100 - t)^3 + C$$

$$Q(100) = C = 0$$

$$Q(t) = -\frac{k}{3}(100 - t)^3$$

$$Q(0) = -\frac{k}{3}(100)^3 = 2{,}000{,}000 \implies k = -6$$

Thus, $Q(t) = 2(100 - t)^3$. When $t = 50$, $Q(50) =$ \$250,000.

112. $\frac{dV}{dt} = \frac{k}{(t + 1)^2}$

$$V(t) = \int \frac{k}{(t + 1)^2}\,dt = -\frac{k}{t + 1} + C$$

$$V(0) = -k + C = 500{,}000$$

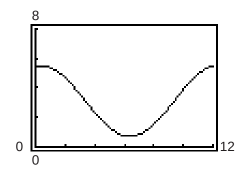
$$V(1) = -\frac{1}{2}k + C = 400{,}000$$

Solving this system yields $k = -200{,}000$ and $C = 300{,}000$. Thus,

$$V(t) = \frac{200{,}000}{t + 1} + 300{,}000.$$

When $t = 4$, $V(4) = \$340{,}000$.

113. $R = 3.121 + 2.399\sin(0.524t + 1.377)$

(a) 

Relative minimum: (6.4, 0.7) or June

Relative maximum: (0.4, 5.5) or January

(b) $\int_0^{12} R(t)\,dt \approx 37.47$ inches

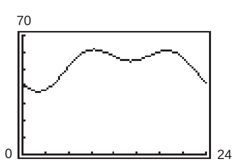
(c) $\frac{1}{3}\int_9^{12} R(t)\,dt \approx \frac{1}{3}(13) = 4.33$ inches

114. $\frac{1}{b - a}\int_a^b \left[74.50 + 43.75\sin\frac{\pi t}{6}\right] dt = \frac{1}{b - a}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_a^b$

(a) $\frac{1}{3}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_0^3 = \frac{1}{3}\left(223.5 + \frac{262.5}{\pi}\right) \approx 102.352$ thousand units

(b) $\frac{1}{3}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_3^6 = \frac{1}{3}\left(447 + \frac{262.5}{\pi} - 223.5\right) \approx 102.352$ thousand units

(c) $\frac{1}{12}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_0^{12} = \frac{1}{12}\left(894 - \frac{262.5}{\pi} + \frac{262.5}{\pi}\right) = 74.5$ thousand units

115. (a) 

Maximum flow: $R \approx 61.713$ at $t = 9.36$.
[(18.861, 61.178) is a relative maximum.]

(b) Volume $= \int_0^{24} R(t)\,dt \approx 1272$ (5 thousands of gallons)

116. $\displaystyle \frac{1}{b-a}\int_a^b [2\sin(60\pi t) + \cos(120\pi t)]\,dt = \frac{1}{b-a}\left[-\frac{1}{30\pi}\cos(60\pi t) + \frac{1}{120\,\pi}\sin(120\pi t)\right]_a^b$

(a) $\displaystyle \frac{1}{(1/60)-0}\left[-\frac{1}{30\pi}\cos(60\pi t) + \frac{1}{120\pi}\sin(120\pi t)\right]_0^{1/60} = 60\left[\left(\frac{1}{30\pi}+0\right) - \left(-\frac{1}{30\pi}\right)\right] = \frac{4}{\pi} \approx 1.273 \text{ amps}$

(b) $\displaystyle \frac{1}{(1/240)-0}\left[-\frac{1}{30\pi}\cos(60\pi t) + \frac{1}{120\pi}\sin(120\pi t)\right]_0^{1/240} = 240\left[\left(-\frac{1}{30\sqrt{2}\pi} + \frac{1}{120\pi}\right) - \left(-\frac{1}{30\pi}\right)\right]$

$\displaystyle = \frac{2}{\pi}\left(5 - 2\sqrt{2}\right) \approx 1.382 \text{ amps}$

(c) $\displaystyle \frac{1}{(1/30)-0}\left[-\frac{1}{30\pi}\cos(60\pi t) + \frac{1}{120\pi}\sin(120\pi t)\right]_0^{1/30} = 30\left[\left(\frac{1}{30\pi}\right) - \left(-\frac{1}{30\pi}\right)\right] = 0 \text{ amps}$

117. $u = 1 - x, x = 1 - u, dx = -du$

When $x = a, u = 1 - a$. When $x = b, u = 1 - b$.

$$P_{a,b} = \int_a^b \frac{15}{4}x\sqrt{1-x}\,dx = \frac{15}{4}\int_{1-a}^{1-b} -(1-u)\sqrt{u}\,du$$

$$= \frac{15}{4}\int_{1-a}^{1-b}(u^{3/2} - u^{1/2})\,du = \frac{15}{4}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_{1-a}^{1-b} = \frac{15}{4}\left[\frac{2u^{3/2}}{15}(3u-5)\right]_{1-a}^{1-b} = \left[-\frac{(1-x)^{3/2}}{2}(3x+2)\right]_a^b$$

(a) $\displaystyle P_{0.50,\,0.75} = \left[-\frac{(1-x)^{3/2}}{2}(3x+2)\right]_{0.50}^{0.75} = 0.353 = 35.3\%$

(b) $\displaystyle P_{0,\,b} = \left[-\frac{(1-x)^{3/2}}{2}(3x+2)\right]_0^b = -\frac{(1-b)^{3/2}}{2}(3b+2) + 1 = 0.5$

$(1-b)^{3/2}(3b+2) = 1$

$b \approx 0.586 = 58.6\%$

118. $u = 1 - x, x = 1 - u, dx = -du$

When $x = a, u = 1 - a$. When $x = b, u = 1 - b$.

$$P_{a,b} = \int_a^b \frac{1155}{32}x^3(1-x)^{3/2}\,dx = \frac{1155}{32}\int_{1-a}^{1-b} -(1-u)^3 u^{3/2}\,du$$

$$= \frac{1155}{32}\int_{1-a}^{1-b}(u^{9/2} - 3u^{7/2} + 3u^{5/2} - u^{3/2})\,du = \frac{1155}{32}\left[\frac{2}{11}u^{11/2} - \frac{2}{3}u^{9/2} + \frac{6}{7}u^{7/2} - \frac{2}{5}u^{5/2}\right]_{1-a}^{1-b}$$

$$= \frac{1155}{32}\left[\frac{2u^{5/2}}{1155}(105u^3 - 385u^2 + 495u - 231)\right]_{1-a}^{1-b} = \left[\frac{u^{5/2}}{16}(105u^3 - 385u^2 + 495u - 231)\right]_{1-a}^{1-b}$$

(a) $\displaystyle P_{0,\,0.25} = \left[\frac{u^{5/2}}{16}(105u^3 - 385u^2 + 495u - 231)\right]_1^{0.75} \approx 0.025 = 2.5\%$

(b) $\displaystyle P_{0.5,\,1} = \left[\frac{u^{5/2}}{16}(105u^3 - 385u^2 + 495u - 231)\right]_{0.5}^{0} \approx 0.736 = 73.6\%$

119. (a) $\displaystyle C = 0.1\int_8^{20}\left[12\sin\frac{\pi(t-8)}{12}\right]dt = \left[-\frac{14.4}{\pi}\cos\frac{\pi(t-8)}{12}\right]_8^{20} = \frac{-14.4}{\pi}(-1-1) \approx \$9.17$

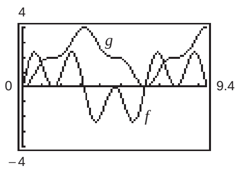
(b) $\displaystyle C = 0.1\int_{10}^{18}\left[12\sin\frac{\pi(t-8)}{12} - 6\right]dt = \left[-\frac{14.4}{\pi}\cos\frac{\pi(t-8)}{12} - 0.6t\right]_{10}^{18}$

$\displaystyle = \left[\frac{-14.4}{\pi}\left(\frac{-\sqrt{3}}{2}\right) - 10.8\right] - \left[\frac{-14.4}{\pi}\left(\frac{\sqrt{3}}{2}\right) - 6\right] \approx \$3.14$

Savings $\approx 9.17 - 3.14 = \$6.03$.

120. $\frac{1}{365}\int_0^{365} 100{,}000\left[1 + \sin\frac{2\pi(t-60)}{365}\right]dt = \frac{100{,}000}{365}\left[t - \frac{365}{2\pi}\cos\frac{2\pi(t-60)}{365}\right]_0^{365} = 100{,}000$ lbs.

121. (a)

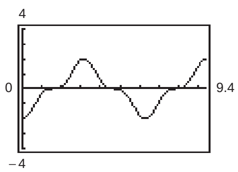


(b) g is nonnegative because the graph of f is positive at the beginning, and generally has more positive sections than negative ones.

(c) The points on g that correspond to the extrema of f are points of inflection of g.

(d) No, some zeros of f, like $x = \pi/2$, do not correspond to an extrema of g. The graph of g continues to increase after $x = \pi/2$ because f remains above the x-axis.

(e) The graph of h is that of g shifted 2 units downward.



$$g(t) = \int_0^t f(x)\,dx$$
$$= \int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^t f(x)\,dx = 2 + h(t).$$

122. Let $f(x) = \sin \pi x$, $0 \le x \le 1$.

Let $\Delta x = \frac{1}{n}$ and use righthand endpoints

$c_i = \frac{i}{n}$, $i = 1, 2, \ldots, n$.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\sin(i\pi/n)}{n} = \lim_{\|\Delta x\|\to 0}\sum_{i=1}^{n} f(c_i)\,\Delta x$$
$$= \int_0^1 \sin \pi x\,dx$$
$$= -\frac{1}{\pi}\cos \pi x\Big]_0^1$$
$$= -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}$$

123. (a) Let $u = 1 - x$, $du = -dx$, $x = 1 - u$

$x = 0 \Rightarrow u = 1$, $x = 1 \Rightarrow u = 0$

$$\int_0^1 x^2(1-x)^5\,dx = \int_1^0 (1-u)^2u^5(-du)$$
$$= \int_0^1 u^5(1-u)^2\,du$$
$$= \int_0^1 x^5(1-x)^2\,dx$$

(b) Let $u = 1 - x$, $du = -dx$, $x = 1 - u$

$x = 0 \Rightarrow u = 1$, $x = 1 \Rightarrow u = 0$

$$\int_0^1 x^a(1-x)^b\,dx = \int_1^0 (1-u)^a u^b(-du)$$
$$= \int_0^1 u^b(1-u)^a\,du$$
$$= \int_0^1 x^b(1-x)^a\,dx$$

124. (a) $\sin x = \cos\left(\frac{\pi}{2} - x\right)$ and $\cos x = \sin\left(\frac{\pi}{2} - x\right)$

Let $u = \frac{\pi}{2} - x$, $du = -dx$, $x = \frac{\pi}{2} - u$:

$$\int_0^{\pi/2}\sin^2 x\,dx = \int_0^{\pi/2}\cos^2\left(\frac{\pi}{2} - x\right)dx = \int_{\pi/2}^0 \cos^2 u(-du)$$
$$= \int_0^{\pi/2}\cos^2 u\,du = \int_0^{\pi/2}\cos^2 x\,dx$$

(b) Let $u = \frac{\pi}{2} - x$ as in part (a):

$$\int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}\cos^n\left(\frac{\pi}{2} - x\right)dx = \int_{\pi/2}^0 \cos^n u(-du)$$
$$= \int_0^{\pi/2}\cos^n u\,du = \int_0^{\pi/2}\cos^n x\,dx$$

125. False

$$\int (2x+1)^2\,dx = \frac{1}{2}\int (2x+1)^2\,2\,dx = \frac{1}{6}(2x+1)^3 + C$$

126. False

$$\int x(x^2+1)^2\,dx = \frac{1}{2}\int (x^2+1)(2x)\,dx = \frac{1}{4}(x^2+1)^2 + C$$

127. True

$$\int_{-10}^{10} (ax^3+bx^2+cx+d)\,dx = \underbrace{\int_{-10}^{10} (ax^3+cx)\,dx}_{\text{Odd}} + \underbrace{\int_{-10}^{10} (bx^2+d)\,dx}_{\text{Even}} = 0 + 2\int_{0}^{10} (bx^2+d)\,dx$$

128. True

$$\int_a^b \sin x\,dx = \Big[-\cos x\Big]_a^b = -\cos b + \cos a = -\cos(b+2\pi) + \cos a = \int_a^{b+2\pi} \sin x\,dx$$

129. True

$$4\int \sin x\cos x\,dx = 2\int \sin 2x\,dx = -\cos 2x + C$$

130. False

$$\int \sin^2 2x\cos 2x\,dx = \frac{1}{2}\int (\sin 2x)^2(2\cos 2x)\,dx = \frac{1}{2}\frac{(\sin 2x)^3}{3} + C = \frac{1}{6}\sin^3 2x + C$$

131. Let $u = cx$, $du = c\,dx$:

$$\begin{aligned} c\int_a^b f(cx)\,dx &= c\int_{ca}^{cb} f(u)\,\frac{du}{c} \\ &= \int_{ca}^{cb} f(u)\,du \\ &= \int_{ca}^{cb} f(x)\,dx \end{aligned}$$

132. (a) $\frac{d}{du}[\sin u - u\cos u + C] = \cos u - \cos u + u\sin u = u\sin u$

Thus, $\int u\sin u\,du = \sin u - u\cos u + C$.

(b) Let $u = \sqrt{x}$, $u^2 = x$, $2\,u\,du = dx$.

$$\begin{aligned} \int_0^{\pi^2} \sin\sqrt{x}\,dx &= \int_0^{\pi} \sin u(2u\,du) \\ &= 2\int_0^{\pi} u\sin u\,du \\ &= 2\Big[\sin u - u\cos u\Big]_0^{\pi} \quad \text{(part (a))} \\ &= 2[-\pi\cos(\pi)] \\ &= 2\pi \end{aligned}$$

133. Because f is odd, $f(-x) = -f(x)$. Then

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$
$$= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx.$$

Let $x = -u$, $dx = -du$ in the first integral.

When $x = 0$, $u = 0$. When $x = -a$, $u = a$.

$$\int_{-a}^{1} f(x)\,dx = -\int_{0}^{a} f(-u)(-du) + \int_{0}^{a} f(x)\,dx$$
$$= -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 0$$

134. Let $u = x + h$, then $du = dx$. When $x = a$, $u = a + h$. When $x = b$, $u = b + h$. Thus,

$$\int_{a}^{b} f(x + h)\,dx = \int_{a+h}^{b+h} f(u)\,du = \int_{a+h}^{b+h} f(x)\,dx.$$

135. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$.

$$\int_{0}^{1} f(x)\,dx = \left[a_0 x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + \cdots + a_n\frac{x^{n+1}}{n+1}\right]_0^1$$
$$= a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0 \quad \text{(Given)}$$

By the Mean Value Theorem for Integrals, there exists c in $[0, 1]$ such that

$$\int_{0}^{1} f(x)\,dx = f(c)(1 - 0)$$
$$0 = f(c).$$

Thus the equation has at least one real zero.

136. $\alpha^2\int_{0}^{1} f(x)\,dx = \alpha^2(1) = \alpha^2$

$$-2\alpha\int_{0}^{1} f(x)x\,dx = -2\alpha(\alpha) = -2\alpha^2$$
$$\int_{0}^{1} f(x)x^2\,dx = \alpha^2$$

Adding,

$$\int_{0}^{1} [\alpha^2 f(x) - 2\alpha\,x f(x) + x^2 f(x)]\,dx = 0$$
$$\int_{0}^{1} f(x)(\alpha - x)^2\,dx = 0.$$

Since $(\alpha - x)^2 \geq 0$, $f = 0$. Hence, there are no such functions.

Section 4.6 Numerical Integration

1. Exact: $\displaystyle\int_{0}^{2} x^2\,dx = \left[\frac{1}{3}x^3\right]_0^2 = \frac{8}{3} \approx 2.6667$

Trapezoidal: $\displaystyle\int_{0}^{2} x^2\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)^2 + 2(1)^2 + 2\left(\frac{3}{2}\right)^2 + (2)^2\right] = \frac{11}{4} = 2.7500$

Simpson's: $\displaystyle\int_{0}^{2} x^2\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^2 + 2(1)^2 + 4\left(\frac{3}{2}\right)^2 + (2)^2\right] = \frac{8}{3} \approx 2.6667$

2. Exact: $\displaystyle\int_{0}^{1} \left(\frac{x^2}{2} + 1\right)dx = \left[\frac{x^3}{6} + x\right]_0^1 = \frac{7}{6} \approx 1.1667$

Trapezoidal: $\displaystyle\int_{0}^{1} \left(\frac{x^2}{2} + 1\right)dx \approx \frac{1}{8}\left[1 + 2\left(\frac{(1/4)^2}{2} + 1\right) + 2\left(\frac{(1/2)^2}{2} + 1\right) + 2\left(\frac{(3/4)^2}{2} + 1\right) + \left(\frac{1^2}{2} + 1\right) = \frac{75}{64} \approx 1.1719\right.$

Simpson's: $\displaystyle\int_{0}^{1} \left(\frac{x^2}{2} + 1\right)dx \approx \frac{1}{12}\left[1 + 4\left(\frac{(1/4)^2}{2} + 1\right) + 2\left(\frac{(1/2)^2}{2} + 1\right) + 4\left(\frac{(3/4)^2}{2} + 1\right) + \left(\frac{1^2}{2} + 1\right)\right] = \frac{7}{6} \approx 1.1667$

3. Exact: $\int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = 4.0000$

Trapezoidal: $\int_0^2 x^3\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)^3 + 2(1)^3 + 2\left(\frac{3}{2}\right)^3 + (2)^3\right] = \frac{17}{4} = 4.2500$

Simpson's: $\int_0^2 x^3\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^3 + 2(1)^3 + 4\left(\frac{3}{2}\right)^3 + (2)^3\right] = \frac{24}{6} = 4.0000$

4. Exact: $\int_1^2 \frac{1}{x^2}\,dx = \left[\frac{-1}{x}\right]_1^2 = 0.5000$

Trapezoidal: $\int_1^2 \frac{1}{x^2}\,dx \approx \frac{1}{8}\left[1 + 2\left(\frac{4}{5}\right)^2 + 2\left(\frac{4}{6}\right)^2 + 2\left(\frac{4}{7}\right)^2 + \frac{1}{4}\right] \approx 0.5090$

Simpson's: $\int_1^2 \frac{1}{x^2}\,dx \approx \frac{1}{12}\left[1 + 4\left(\frac{4}{5}\right)^2 + 2\left(\frac{4}{6}\right)^2 + 4\left(\frac{4}{7}\right)^2 + \frac{1}{4}\right] \approx 0.5004$

5. Exact: $\int_0^2 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^2 = 4.0000$

Trapezoidal: $\int_0^2 x^3\,dx \approx \frac{1}{8}\left[0 + 2\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 2\left(\frac{3}{4}\right)^3 + 2(1)^3 + 2\left(\frac{5}{4}\right)^3 + 2\left(\frac{6}{4}\right)^3 + 2\left(\frac{7}{4}\right)^3 + 8\right] = 4.0625$

Simpson's: $\int_0^2 x^3\,dx \approx \frac{1}{12}\left[0 + 4\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 4\left(\frac{3}{4}\right)^3 + 2(1)^3 + 4\left(\frac{5}{4}\right)^3 + 2\left(\frac{6}{4}\right)^3 + 4\left(\frac{7}{4}\right)^3 + 8\right] = 4.0000$

6. Exact: $\int_0^8 \sqrt[3]{x}\,dx = \left[\frac{3}{4}x^{4/3}\right]_0^8 = 12.0000$

Trapezoidal: $\int_0^8 \sqrt[3]{x}\,dx \approx \frac{1}{2}[0 + 2 + 2\sqrt[3]{2} + 2\sqrt[3]{3} + 2\sqrt[3]{4} + 2\sqrt[3]{5} + 2\sqrt[3]{6} + 2\sqrt[3]{7} + 2] \approx 11.7296$

Simpson's: $\int_0^8 \sqrt[3]{x}\,dx \approx \frac{1}{3}[0 + 4 + 2\sqrt[3]{2} + 4\sqrt[3]{3} + 2\sqrt[3]{4} + 4\sqrt[3]{5} + 2\sqrt[3]{6} + 4\sqrt[3]{7} + 2] \approx 11.8632$

7. Exact: $\int_4^9 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_4^9 = 18 - \frac{16}{3} = \frac{38}{3} \approx 12.6667$

Trapezoidal: $\int_4^9 \sqrt{x}\,dx \approx \frac{5}{16}\left[2 + 2\sqrt{\frac{37}{8}} + 2\sqrt{\frac{21}{4}} + 2\sqrt{\frac{47}{8}} + 2\sqrt{\frac{26}{4}} + 2\sqrt{\frac{57}{8}} + 2\sqrt{\frac{31}{4}} + 2\sqrt{\frac{67}{8}} + 3\right]$

≈ 12.6640

Simpson's: $\int_4^9 \sqrt{x}\,dx \approx \frac{5}{24}\left[2 + 4\sqrt{\frac{37}{8}} + \sqrt{21} + 4\sqrt{\frac{47}{8}} + \sqrt{26} + 4\sqrt{\frac{57}{8}} + \sqrt{31} + 4\sqrt{\frac{67}{8}} + 3\right] \approx 12.6667$

8. Exact: $\int_1^3 (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_1^3 = 3 - \frac{11}{3} = -\frac{2}{3} \approx -0.6667$

Trapezoidal: $\int_1^3 (4 - x^2)\,dx \approx \frac{1}{4}\left\{3 + 2\left[4 - \left(\frac{3}{2}\right)^2\right] + 2(0) + 2\left[4 - \left(\frac{5}{2}\right)^2\right] - 5\right\} = -0.7500$

Simpson's: $\int_1^3 (4 - x^2)\,dx \approx \frac{1}{6}\left[3 + 4\left(4 - \frac{9}{4}\right) + 0 + 4\left(4 - \frac{25}{4}\right) - 5\right] \approx -0.6667$

9. Exact: $\int_1^2 \frac{1}{(x+1)^2}\,dx = \left[-\frac{1}{x+1}\right]_1^2 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6} \approx 0.1667$

Trapezoidal: $\int_1^2 \frac{1}{(x+1)^2}\,dx \approx \frac{1}{8}\left[\frac{1}{4} + 2\left(\frac{1}{((5/4)+1)^2}\right) + 2\left(\frac{1}{((3/2)+1)^2}\right) + 2\left(\frac{1}{((7/4)+1)^2}\right) + \frac{1}{9}\right]$

$= \frac{1}{8}\left(\frac{1}{4} + \frac{32}{81} + \frac{8}{25} + \frac{32}{121} + \frac{1}{9}\right) \approx 0.1676$

Simpson's: $\int_1^2 \frac{1}{(x+1)^2}\,dx \approx \frac{1}{12}\left[\frac{1}{4} + 4\left(\frac{1}{((5/4)+1)^2}\right) + 2\left(\frac{1}{((3/2)+1)^2}\right) + 4\left(\frac{1}{((7/4)+1)^2}\right) + \frac{1}{9}\right]$

$= \frac{1}{12}\left(\frac{1}{4} + \frac{64}{81} + \frac{8}{25} + \frac{64}{121} + \frac{1}{9}\right) \approx 0.1667$

10. Exact: $\int_0^2 x\sqrt{x^2+1}\,dx = \frac{1}{3}\left[(x^2+1)^{3/2}\right]_0^2 = \frac{1}{3}(5^{3/2} - 1) \approx 3.393$

Trapezoidal: $\int_0^2 x\sqrt{x^2+1}\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)\sqrt{(1/2)^2+1} + 2(1)\sqrt{1^2+1} + 2\left(\frac{3}{2}\right)\sqrt{(3/2)^2+1} + 2\sqrt{2^2+1}\right] \approx 3.457$

Simpson's: $\int_0^2 x\sqrt{x^2+1}\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)\sqrt{(1/2)^2+1} + 2(1)\sqrt{1^2+1} + 4\left(\frac{3}{2}\right)\sqrt{(3/2)^2+1} + 2\sqrt{2^2+1}\right] \approx 3.392$

11. Trapezoidal: $\int_0^2 \sqrt{1+x^3}\,dx \approx \frac{1}{4}[1 + 2\sqrt{1+(1/8)} + 2\sqrt{2} + 2\sqrt{1+(27/8)} + 3] \approx 3.283$

Simpson's: $\int_0^2 \sqrt{1+x^3}\,dx \approx \frac{1}{6}[1 + 4\sqrt{1+(1/8)} + 2\sqrt{2} + 4\sqrt{1+(27/8)} + 3] \approx 3.240$

Graphing utility: 3.241

12. Trapezoidal: $\int_0^2 \frac{1}{\sqrt{1+x^3}}\,dx \approx \frac{1}{4}\left[1 + 2\left(\frac{1}{\sqrt{1+(1/2)^3}}\right) + 2\left(\frac{1}{\sqrt{1+1^3}}\right) + 2\left(\frac{1}{\sqrt{1+(3/2)^3}}\right) + \frac{1}{3}\right] \approx 1.397$

Simpson's: $\int_0^2 \frac{1}{\sqrt{1+x^3}}\,dx \approx \frac{1}{6}\left[1 + 4\left(\frac{1}{\sqrt{1+(1/2)^3}}\right) + 2\left(\frac{1}{\sqrt{1+1^3}}\right) + 4\left(\frac{1}{\sqrt{1+(3/2)^3}}\right) + \frac{1}{3}\right] \approx 1.405$

Graphing utility: 1.402

13. $\int_0^1 \sqrt{x}\sqrt{1-x}\,dx = \int_0^1 \sqrt{x(1-x)}\,dx$

Trapezoidal: $\int_0^1 \sqrt{x(1-x)}\,dx \approx \frac{1}{8}\left[0 + 2\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 2\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)}\right] \approx 0.342$

Simpson's: $\int_0^1 \sqrt{x(1-x)}\,dx \approx \frac{1}{12}\left[0 + 4\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 4\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)}\right] \approx 0.372$

Graphing utility: 0.393

14. Trapezoidal: $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx \approx \frac{\pi}{16}\left[\sqrt{\frac{\pi}{2}}(1) + 2\sqrt{\frac{5\pi}{8}}\sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}}\sin\left(\frac{3\pi}{4}\right) + 2\sqrt{\frac{7\pi}{8}}\sin\left(\frac{7\pi}{8}\right) + 0\right] \approx 1.430$

Simpson's: $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx \approx \frac{\pi}{24}\left[\sqrt{\frac{\pi}{2}} + 4\sqrt{\frac{5\pi}{8}}\sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}}\sin\left(\frac{3\pi}{4}\right) + 4\sqrt{\frac{7\pi}{8}}\sin\left(\frac{7\pi}{8}\right) + 0\right] \approx 1.458$

Graphing utility: 1.458

15. Trapezoidal: $\int_0^{\sqrt{\pi/2}} \cos(x^2)\,dx \approx \frac{\sqrt{\pi/2}}{8}\left[\cos 0 + 2\cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2\cos\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 2\cos\left(\frac{3\sqrt{\pi/2}}{4}\right)^2 + \cos\left(\sqrt{\frac{\pi}{2}}\right)^2\right]$

≈ 0.957

Simpson's: $\int_0^{\sqrt{\pi/2}} \cos(x^2)\,dx \approx \frac{\sqrt{\pi/2}}{12}\left[\cos 0 + 4\cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2\cos\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 4\cos\left(\frac{3\sqrt{\pi/2}}{4}\right)^2 + \cos\left(\sqrt{\frac{\pi}{2}}\right)^2\right]$

≈ 0.978

Graphing utility: 0.977

16. Trapezoidal: $\int_0^{\sqrt{\pi/4}} \tan(x^2)\,dx \approx \frac{\sqrt{\pi/4}}{8}\left[\tan 0 + 2\tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2\tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 2\tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right]$

≈ 0.271

Simpson's: $\int_0^{\sqrt{\pi/4}} \tan(x^2)\,dx \approx \frac{\sqrt{\pi/4}}{12}\left[\tan 0 + 4\tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2\tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 4\tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right]$

≈ 0.257

Graphing utility: 0.256

17. Trapezoidal: $\int_1^{1.1} \sin x^2\,dx \approx \frac{1}{80}[\sin(1) + 2\sin(1.025)^2 + 2\sin(1.05)^2 + 2\sin(1.075)^2 + \sin(1.1)^2] \approx 0.089$

Simpson's: $\int_1^{1.1} \sin x^2\,dx \approx \frac{1}{120}[\sin(1) + 4\sin(1.025)^2 + 2\sin(1.05)^2 + 4\sin(1.075)^2 + \sin(1.1)^2] \approx 0.089$

Graphing utility: 0.089

18. Trapezoidal: $\int_0^{\pi/2} \sqrt{1+\cos^2 x}\,dx \approx \frac{\pi}{16}\left[\sqrt{2} + 2\sqrt{1+\cos^2(\pi/8)} + 2\sqrt{1+\cos^2(\pi/4)} + 2\sqrt{1+\cos^2(3\pi/8)} + 1\right] \approx 1.910$

Simpson's: $\int_0^{\pi/2} \sqrt{1+\cos^2 x}\,dx \approx \frac{\pi}{24}\left[\sqrt{2} + 4\sqrt{1+\cos^2(\pi/8)} + 2\sqrt{1+\cos^2(\pi/4)} + 4\sqrt{1+\cos^2(3\pi/8)} + 1\right] \approx 1.910$

Graphing utility: 1.910

19. Trapezoidal: $\int_0^{\pi/4} x\tan x\,dx \approx \frac{\pi}{32}\left[0 + 2\left(\frac{\pi}{16}\right)\tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right)\tan\left(\frac{2\pi}{16}\right) + 2\left(\frac{3\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4}\right] \approx 0.194$

Simpson's: $\int_0^{\pi/4} x\tan x\,dx \approx \frac{\pi}{48}\left[0 + 4\left(\frac{\pi}{16}\right)\tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right)\tan\left(\frac{2\pi}{16}\right) + 4\left(\frac{3\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4}\right] \approx 0.186$

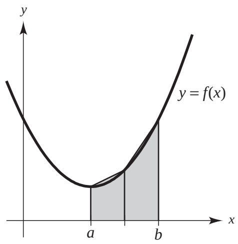
Graphing utility: 0.186

20. Trapezoidal: $\int_0^{\pi} \frac{\sin x}{x}\,dx \approx \frac{\pi}{8}\left[1 + \frac{2\sin(\pi/4)}{\pi/4} + \frac{2\sin(\pi/2)}{\pi/2} + \frac{2\sin(3\pi/4)}{3\pi/4} + 0\right] \approx 1.836$

Simpson's: $\int_0^{\pi} \frac{\sin x}{x}\,dx \approx \frac{\pi}{12}\left[1 + \frac{4\sin(\pi/4)}{\pi/4} + \frac{2\sin(\pi/2)}{\pi/2} + \frac{4\sin(3\pi/4)}{3\pi/4} + 0\right] \approx 1.852$

Graphing utility: 1.852

21.



The Trapezoidal Rule overestimates the area if the graph of the integrand is concave up.

22. Trapezoidal: Linear polynomials

Simpson's: Quadratic polynomials

23. $f(x) = x^3$

$f'(x) = 3x^2$

$f''(x) = 6x$

$f'''(x) = 6$

$f^{(4)}(x) = 0$

(a) Trapezoidal: Error $\leq \dfrac{(2-0)^3}{12(4^2)}(12) = 0.5$ since

$|f''(x)|$ is maximum in $[0, 2]$ when $x = 2$.

(b) Simpson's: Error $\leq \dfrac{(2-0)^5}{180(4^4)}(0) = 0$ since

$f^{(4)}(x) = 0$.

24. $f(x) = 2x + 3$

$f'(x) = 2$

$f''(x) = 0$

The error is 0 for both rules.

25. $f(x) = \dfrac{1}{x+1}$

$f'(x) = \dfrac{-1}{(x+1)^2}$

$f''(x) = \dfrac{2}{(x+1)^3}$

$f'''(x) = \dfrac{-6}{(x+1)^4}$

$f^{(4)}(x) = \dfrac{24}{(x+1)^5}$

(a) Trapezoidal: Error $\leq \dfrac{(1-0)^2}{12(4^2)}(2) = \dfrac{1}{96} \approx 0.01$ since

$f''(x)$ is maximum in $[0, 1]$ when $x = 0$.

(b) Simpson's: Error $\leq \dfrac{(1-0)^5}{180(4^4)}(24) = \dfrac{1}{1920} \approx 0.0005$

since $f^{(4)}(x)$ is maximum in $[0, 1]$ when $x = 0$.

26. $f(x) = (x-1)^{-2}$

$f'(x) = -2(x-1)^{-3}$

$f''(x) = 6(x-1)^{-4}$

$f'''(x) = -24(x-1)^{-5}$

$f^{(4)}(x) = 120(x-1)^{-6}$

(a) Trapezoidal: Error $\leq \dfrac{(4-2)^3}{12(4^2)}(6) = \dfrac{1}{4}$,

since $|f''(x)|$ is a maximum of 6 at $x = 2$.

(b) Simpson's: Error $\leq \dfrac{(4-2)^5}{180(4^4)}(120) = \dfrac{1}{12}$

since $|f^{(4)}(x)|$ is a maximum of 120 at $x = 2$.

27. $f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{(4)}(x) = \cos x$

(a) Trapezoidal: Error $\leq \frac{(\pi - 0)^3}{12(4^2)}(1) = \frac{\pi^3}{192} \approx 0.1615$

because $|f''(x)|$ is at most 1 on $[0, \pi]$.

(b) Simpson's: Error $\leq \frac{(\pi - 0)^5}{180(4^4)}(1) = \frac{\pi^5}{46{,}080} \approx 0.006641$

because $|f^{(4)}(x)|$ is at most 1 on $[0, \pi]$.

28. $f(x) = \sin(\pi x)$

$f'(x) = \pi\cos(\pi x)$

$f''(x) = -\pi^2 \sin(\pi x)$

$f'''(x) = -\pi^3 \cos(\pi x)$

$f^{(4)}(x) = \pi^4 \sin(\pi x)$

(a) Trapezoidal: Error $\leq \frac{(1 - 0)^3}{12(4^2)}\pi^2 = \frac{\pi^2}{192} \approx 0.0514$,

since $|f''(x)| \leq \pi^2$ on $[0, 1]$.

(b) Simpson's: Error $\leq \frac{(1 - 0)^5}{180(4^4)}\pi^4 = \frac{\pi^4}{46{,}080} \approx 0.0021$,

since $|f^{(4)}(x)| \leq \pi^4$ on $[0, 1]$.

29. $f''(x) = \frac{2}{x^3}$ in $[1, 3]$.

(a) $|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| = 2$.

Trapezoidal: Error $\leq \frac{2^3}{12n^2}(2) < 0.00001$, $n^2 > 133{,}333.33$, $n > 365.15$; let $n = 366$.

$f^{(4)}(x) = \frac{24}{x^5}$ in $[1, 3]$.

(b) $|f^{(4)}(x)|$ is maximum when $x = 1$ and $|f^{(4)}(1)| = 24$.

Simpson's: Error $\leq \frac{2^5}{180n^4}(24) < 0.00001$, $n^4 > 426{,}666.67$, $n > 25.56$; let $n = 26$.

30. $f''(x) = \frac{2}{(1 + x)^3}$ in $[0, 1]$.

(a) $|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = 2$.

Trapezoidal: Error $\leq \frac{1}{12n^2}(2) < 0.00001$, $n^2 > 16{,}666.67$, $n > 129.10$; let $n = 130$.

$f^{(4)}(x) = \frac{24}{(1 + x)^5}$ in $[0, 1]$

(b) $|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = 24$.

Simpson's: Error $\leq \frac{1}{180n^4}(24) < 0.00001$, $n^4 > 13{,}333.33$, $n > 10.75$; let $n = 12$. (In Simpson's Rule n must be even.)

31. $f(x) = (x+2)^{1/2}, \quad 0 \le x \le 2$

$$f'(x) = \frac{1}{2}(x+2)^{-1/2}$$

$$f''(x) = -\frac{1}{4}(x+2)^{-3/2}$$

$$f'''(x) = \frac{3}{8}(x+2)^{-5/2}$$

$$f^{(4)}(x) = \frac{-15}{16}(x+2)^{-7/2}$$

(a) Maximum of $|f''(x)| = \left|\frac{-1}{4(x+2)^{3/2}}\right|$ is $\frac{\sqrt{2}}{16} \approx 0.0884$.

Trapezoidal: $\text{Error} \le \frac{(2-0)^3}{12n^2}\left(\frac{\sqrt{2}}{16}\right) \le 0.00001$

$$n^2 \ge \frac{8\sqrt{2}}{12(16)}10^5 = \frac{\sqrt{2}}{24}10^5$$

$n \ge 76.8$. Let $n = 77$.

(b) Maximum of $|f^{(4)}(x)| = \left|\frac{-15}{16(x+2)^{7/2}}\right|$ is

$$\frac{15\sqrt{2}}{256} \approx 0.0829.$$

Simpson's: $\text{Error} \le \frac{2^5}{180n^4}\left(\frac{15\sqrt{2}}{256}\right) \le 0.00001$

$$n^4 \ge \frac{32(15)\sqrt{2}}{180(256)}10^5 = \frac{\sqrt{2}}{96}10^5$$

$n \ge 6.2$. Let $n = 8$ (even).

32. $f(x) = x^{-1/2}, \quad 1 \le x \le 3$

$$f'(x) = \frac{-1}{2}x^{-3/2}$$

$$f''(x) = \frac{3}{4}x^{-5/2}$$

$$f'''(x) = \frac{-15}{8}x^{-7/2}$$

$$f^{(4)}(x) = \frac{105}{16}x^{-9/2}$$

(a) Maximum of $|f''(x)| = \left|\frac{3}{4x^{5/2}}\right|$ is $\frac{3}{4}$ on $[1, 3]$.

Trapezoidal: $\text{Error} \le \frac{(3-1)^3}{12n^2}\left(\frac{3}{4}\right) \le 0.00001$

$$n^2 \ge \frac{1}{2} \cdot 10^5$$

$n \ge 223.6$. Let $n = 224$

(b) Maximum of $|f^{(4)}(x)| = \left|\frac{105}{16x^{9/2}}\right|$ is $\frac{105}{16}$ on $[1, 3]$.

Simpson's: $\text{Error} \le \frac{2^5}{180n^4}\left(\frac{105}{16}\right) \le 0.00001$

$$n^4 \ge \frac{7}{6} \cdot 10^5$$

$n \ge 18.5$. Let $n = 20$ (even)

33. $f(x) = \cos(\pi x), \quad 0 \le x \le 1$

$$f'(x) = -\pi \sin(\pi x)$$

$$f''(x) = -\pi^2 \cos(\pi x)$$

$$f'''(x) = \pi^3 \sin(\pi x)$$

$$f^{(4)}(x) = \pi^4 \cos(\pi x)$$

(a) Maximum of $|f''(x)| = |-\pi^2 \cos(\pi x)|$ is π^2.

Trapezoidal: $\text{Error} \le \frac{(1-0)^3}{12n^2}\pi^2 \le 0.00001$

$$n^2 \ge \frac{\pi^2}{12} \cdot 10^5$$

$n \ge 286.8$. Let $n = 287$.

(b) Maximum of $|f^{(4)}(x)| = |\pi^4 \cos(\pi x)|$ is π^4.

Simpson's: $\text{Error} \le \frac{1}{180n^4}\pi^4 \le 0.00001$

$$n^4 \ge \frac{\pi^4}{180} \cdot 10^5$$

$n \ge 15.3$. Let $n = 16$.

34. $f(x) = \sin x, \quad 0 \le x \le \frac{\pi}{2}$

$$f'(x) = \cos x$$

$$f''(x) = -\sin x$$

$$f'''(x) = -\cos x$$

$$f^{(4)}(x) = \sin x$$

All derivatives are bounded by 1.

(a) Trapezoidal: $\text{Error} \le \frac{(\pi/2)^3}{12n^2}(1) \le 0.00001$

$$n^2 \ge \frac{\pi^3}{96} \cdot 10^5$$

$n \ge 179.7$. Let $n = 180$.

(b) Simpson's: $\text{Error} \le \frac{(\pi/2)^5}{180n^4}(1) \le 0.00001$

$$n^4 \ge \frac{\pi^5}{5760} \cdot 10^5$$

$n \ge 8.5$. Let $n = 10$ (even).

35. $f(x) = \sqrt{1 + x}$

(a) $f''(x) = -\dfrac{1}{4(1 + x)^{3/2}}$ in $[0, 2]$.

$|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = \dfrac{1}{4}$.

Trapezoidal: Error $\leq \dfrac{8}{12n^2}\left(\dfrac{1}{4}\right) < 0.00001$, $n^2 > 16{,}666.67$, $n > 129.10$; let $n = 130$.

(b) $f^{(4)}(x) = \dfrac{-15}{16(1 + x)^{7/2}}$ in $[0, 2]$

$|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = \dfrac{15}{16}$.

Simpson's: Error $\leq \dfrac{32}{180n^4}\left(\dfrac{15}{16}\right) < 0.00001$, $n^4 > 16{,}666.67$, $n > 11.36$; let $n = 12$.

36. $f(x) = (x + 1)^{2/3}$

(a) $f''(x) = -\dfrac{2}{9(x + 1)^{4/3}}$ in $[0, 2]$.

$|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = \dfrac{2}{9}$.

Trapezoidal: Error $\leq \dfrac{8}{12n^4}\left(\dfrac{2}{9}\right) < 0.00001$, $n^2 > 14{,}814.81$, $n > 121.72$; let $n = 122$.

(b) $f^{(4)}(x) = -\dfrac{56}{81(x + 1)^{10/3}}$ in $[0, 2]$.

$|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = \dfrac{56}{81}$.

Simpson's: Error $\leq \dfrac{32}{180n^4}\left(\dfrac{56}{81}\right) < 0.00001$, $n^4 > 12{,}290.81$, $n > 10.53$; let $n = 12$. (In Simpson's Rule n must be even.)

37. $f(x) = \tan(x^2)$

(a) $f''(x) = 2\sec^2(x^2)[1 + 4x^2\tan(x^2)]$ in $[0, 1]$.

$|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| \approx 49.5305$.

Trapezoidal: Error $\leq \dfrac{(1 - 0)^3}{12n^2}(49.5305) < 0.00001$, $n^2 > 412{,}754.17$, $n > 642.46$; let $n = 643$.

(b) $f^{(4)}(x) = 8\sec^2(x^2)[12x^2 + (3 + 32x^4)\tan(x^2) + 36x^2\tan^2(x^2) + 48x^4\tan^3(x^2)]$ in $[0, 1]$

$|f^{(4)}(x)|$ is maximum when $x = 1$ and $|f^{(4)}(1)| \approx 9184.4734$.

Simpson's: Error $\leq \dfrac{(1 - 0)^5}{180n^4}(9184.4734) < 0.00001$, $n^4 > 5{,}102{,}485.22$, $n > 47.53$; let $n = 48$.

38. $f(x) = \sin(x^2)$

(a) $f''(x) = 2[-2x^2\sin(x^2) + \cos(x^2)]$ in $[0, 1]$.

$|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| \approx 2.2853$.

Trapezoidal: Error $\leq \dfrac{(1 - 0)^3}{12n^2}(2.2853) < 0.00001$, $n^2 > 19{,}044.17$, $n > 138.00$; let $n = 139$.

(b) $f^{(4)}(x) = (16x^4 - 12)\sin(x^2) - 48x^2\cos(x^2)$ in $[0, 1]$

$|f^{(4)}(x)|$ is maximum when $x \approx 0.852$ and $|f^{(4)}(0.852)| \approx 28.4285$.

Simpson's: Error $\leq \dfrac{(1 - 0)^5}{180n^4}(28.4285) < 0.00001$, $n^4 > 15{,}793.61$, $n > 11.21$; let $n = 12$.

39. (a) $b - a = 4 - 0 = 4, \quad n = 4$

$$\int_0^4 f(x)\,dx \approx \frac{4}{8}[3 + 2(7) + 2(9) + 2(7) + 0]$$

$$= \frac{1}{2}(49) = \frac{49}{2} = 24.5$$

(b) $$\int_0^4 f(x)\,dx \approx \frac{4}{12}[3 + 4(7) + 2(9) + 4(7) + 0]$$

$$= \frac{77}{3} \approx 25.67$$

40. $n = 8, b - a = 8 - 0 = 8$

(a) $$\int_0^8 f(x)\,dx \approx \frac{8}{16}[0 + 2(1.5) + 2(3) + 2(5.5) + 2(9) + 2(10) + 2(9) + 2(6) + 0]$$

$$= \frac{1}{2}(88) = 44$$

(b) $$\int_0^8 f(x)\,dx \approx \frac{8}{24}[0 + 4(1.5) + 2(3) + 4(5.5) + 2(9) + 4(10) + 2(9) + 4(6) + 0]$$

$$= \frac{1}{3}(134) = \frac{134}{3}$$

41. The program will vary depending upon the computer or programmable calculator that you use.

42. $f(x) = \sqrt{2 + 3x^2}$ on $[0, 4]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	12.7771	15.3965	18.4340	15.6055	15.4845
8	14.0868	15.4480	16.9152	15.5010	15.4662
10	14.3569	15.4544	16.6197	15.4883	15.4658
12	14.5386	15.4578	16.4242	15.4814	15.4657
16	14.7674	15.4613	16.1816	15.4745	15.4657
20	14.9056	15.4628	16.0370	15.4713	15.4657

43. $f(x) = \sqrt{1 - x^2}$ on $[0, 1]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	0.8739	0.7960	0.6239	0.7489	0.7709
8	0.8350	0.7892	0.7100	0.7725	0.7803
10	0.8261	0.7881	0.7261	0.7761	0.7818
12	0.8200	0.7875	0.7367	0.7783	0.7826
16	0.8121	0.7867	0.7496	0.7808	0.7836
20	0.8071	0.7864	0.7571	0.7821	0.7841

44. $f(x) = \sin\sqrt{x}$ on $[0, 4]$.

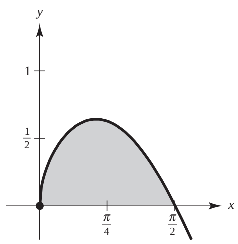
n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	2.8163	3.5456	3.7256	3.2709	3.3996
8	3.1809	3.5053	3.6356	3.4083	3.4541
10	3.2478	3.4990	3.6115	3.4296	3.4624
12	3.2909	3.4952	3.5940	3.4425	3.4674
16	3.3431	3.4910	3.5704	3.4568	3.4730
20	3.3734	3.4888	3.5552	3.4643	3.4759

45. $A = \int_0^{\pi/2} \sqrt{x}\cos x\,dx$

Simpson's Rule: $n = 14$

$$\int_0^{\pi/2} \sqrt{x}\cos x\,dx \approx \frac{\pi}{84}\left[\sqrt{0}\cos 0 + 4\sqrt{\frac{\pi}{28}}\cos\frac{\pi}{28} + 2\sqrt{\frac{\pi}{14}}\cos\frac{\pi}{14} + 4\sqrt{\frac{3\pi}{28}}\cos\frac{3\pi}{28} + \cdots + \sqrt{\frac{\pi}{2}}\cos\frac{\pi}{2}\right]$$

$$\approx 0.701$$



46. Simpson's Rule: $n = 8$

$$8\sqrt{3}\int_0^{\pi/2}\sqrt{1 - \frac{2}{3}\sin^2\theta}\,d\theta \approx \frac{\sqrt{3}\pi}{6}\left[\sqrt{1 - \frac{2}{3}\sin^2 0} + 4\sqrt{1 - \frac{2}{3}\sin^2\frac{\pi}{16}} + 2\sqrt{1 - \frac{2}{3}\sin^2\frac{\pi}{8}} + \cdots + \sqrt{1 - \frac{2}{3}\sin^2\frac{\pi}{2}}\right]$$

$$\approx 17.476$$

47. $W = \int_0^5 100x\sqrt{125 - x^3}\,dx$

Simpson's Rule: $n = 12$

$$\int_0^5 100x\sqrt{125 - x^3}\,dx \approx \frac{5}{3(12)}\left[0 + 400\left(\frac{5}{12}\right)\sqrt{125 - \left(\frac{5}{12}\right)^3} + 200\left(\frac{10}{12}\right)\sqrt{125 - \left(\frac{10}{12}\right)^3} + 400\left(\frac{15}{12}\right)\sqrt{125 - \left(\frac{15}{12}\right)^3} + \cdots + 0\right] \approx 10{,}233.58 \text{ ft} \cdot \text{lb}$$

48. (a) Trapezoidal:

$$\int_0^2 f(x)\,dx \approx \frac{2}{2(8)}[4.32 + 2(4.36) + 2(4.58) + 2(5.79) + 2(6.14) + 2(7.25) + 2(7.64) + 2(8.08) + 8.14] \approx 12.518$$

Simpson's:

$$\int_0^2 f(x)\,dx \approx \frac{2}{3(8)}[4.32 + 4(4.36) + 2(4.58) + 4(5.79) + 2(6.14) + 4(7.25) + 2(7.64) + 4(8.08) + 8.14] \approx 12.592$$

(b) Using a graphing utility,

$y = -1.3727x^3 + 4.0092x^2 - 0.6202x + 4.2844.$

Integrating, $\int_0^2 y\,dx \approx 12.53.$

49. $\int_0^{1/2} \frac{6}{\sqrt{1 - x^2}}\,dx$ Simpson's Rule, $n = 6$

$$\pi \approx \frac{\left(\frac{1}{2} - 0\right)}{3(6)}[6 + 4(6.0209) + 2(6.0851) + 4(6.1968) + 2(6.3640) + 4(6.6002) + 6.9282]$$

$$\approx \frac{1}{36}[113.098] \approx 3.1416$$

50. Simpson's Rule: $n = 6$

$$\pi = 4\int_0^1 \frac{1}{1+x^2}\,dx \approx \frac{4}{3(6)}\left[1 + \frac{4}{1+(1/6)^2} + \frac{2}{1+(2/6)^2} + \frac{4}{1+(3/6)^2} + \frac{2}{1+(4/6)^2} + \frac{4}{1+(5/6)^2} + \frac{1}{2}\right]$$

$$\approx 3.14159$$

51. Area $\approx \frac{1000}{2(10)}[125 + 2(125) + 2(120) + 2(112) + 2(90) + 2(90) + 2(95) + 2(88) + 2(75) + 2(35)] = 89{,}250$ sq m

52. Area $\approx \frac{120}{2(12)}[75 + 2(81) + 2(84) + 2(76) + 2(67) + 2(68) + 2(69) + 2(72) + 2(68) + 2(56) + 2(42) + 2(23) + 0]$

$= 7435$ sq m

53. Let $f(x) = Ax^3 + Bx^2 + Cx + D$. Then $f^{(4)}(x) = 0$.

Simpson's: Error $\leq \frac{(b-a)^5}{180n^4}(0) = 0$

Therefore, Simpson's Rule is exact when approximating the integral of a cubic polynomial.

Example: $\int_0^1 x^3\,dx = \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^3 + 1\right] = \frac{1}{4}$

This is the exact value of the integral.

54. $\int_0^t \sin\sqrt{x}\,dx = 2,\ n = 10$

By trial and error, we obtain $t \approx 2.477$.

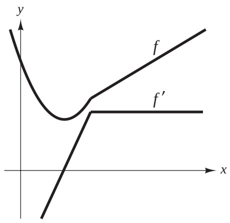
55. The quadratic polynomial

$$p(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}y_3$$

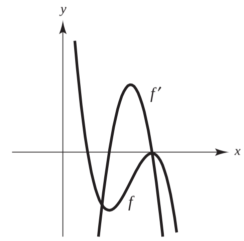
passes through the three points.

Review Exercises for Chapter 4

1.



2.



3. $\int (2x^2 + x - 1)\,dx = \frac{2}{3}x^3 + \frac{1}{2}x^2 - x + C$

4. $\int \frac{2}{\sqrt[3]{3x}}\,dx = \frac{2}{\sqrt[3]{3}}\int x^{-1/3}dx = \frac{2}{\sqrt[3]{3}}\frac{x^{2/3}}{(2/3)} + C$

$= \frac{3}{\sqrt[3]{3}}x^{2/3} + C$

$= (3x)^{2/3} + C$

5. $\int \frac{x^3+1}{x^2}\,dx = \int \left(x + \frac{1}{x^2}\right) dx = \frac{1}{2}x^2 - \frac{1}{x} + C$

6. $\int \frac{x^3 - 2x^2 + 1}{x^2}\,dx = \int (x - 2 + x^{-2})\,dx$

$$= \frac{1}{2}x^2 - 2x - \frac{1}{x} + C$$

7. $\int (4x - 3\sin x)\,dx = 2x^2 + 3\cos x + C$

8. $\int (5\cos x - 2\sec^2 x)\,dx = 5\sin x - 2\tan x + C$

9. $f'(x) = -2x,\ (-1, 1)$

$f(x) = \int -2x\,dx = -x^2 + C$

When $x = -1$:

$y = -1 + C = 1$

$C = 2$

$y = 2 - x^2$

10. $f''(x) = 6(x - 1)$

$f'(x) = \int 6(x-1)\,dx = 3(x-1)^2 + C_1$

Since the slope of the tangent line at $(2, 1)$ is 3, it follows that $f'(2) = 3 + C_1 = 3$ when $C_1 = 0$.

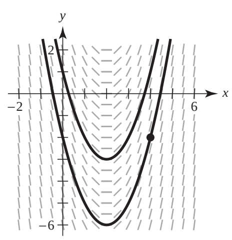
$f'(x) = 3(x-1)^2$

$f(x) = \int 3(x-1)^2\,dx = (x-1)^3 + C_2$

$f(2) = 1 + C_2 = 1$ when $C_2 = 0$.

$f(x) = (x-1)^3$

11. (a)

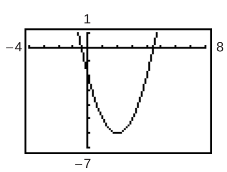


(b) $\frac{dy}{dx} = 2x - 4, \quad (4, -2)$

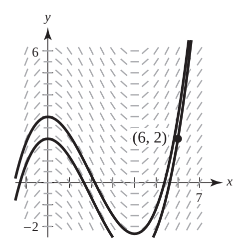
$y = \int (2x - 4)\,dx = x^2 - 4x + C$

$-2 = 16 - 16 + C \implies C = -2$

$y = x^2 - 4x - 2$



12. (a)



(b) $\frac{dy}{dx} = \frac{1}{2}x^2 - 2x, \quad (6, 2)$

$y = \int \left(\frac{1}{2}x^2 - 2x\right) dx = \frac{1}{6}x^3 - x^2 + C$

$2 = \frac{1}{6}(6^3) - 6^2 + C \implies C = 2$

$y = \frac{1}{6}x^3 - x^2 + 2$

13. $a(t) = a$

$$v(t) = \int a\,dt = at + C_1$$

$v(0) = 0 + C_1 = 0$ when $C_1 = 0$.

$$v(t) = at$$

$$s(t) = \int at\,dt = \frac{a}{2}t^2 + C_2$$

$s(0) = 0 + C_2 = 0$ when $C_2 = 0$.

$$s(t) = \frac{a}{2}t^2$$

$$s(30) = \frac{a}{2}(30)^2 = 3600 \text{ or}$$

$$a = \frac{2(3600)}{(30)^2} = 8 \text{ ft/sec}^2.$$

$$v(30) = 8(30) = 240 \text{ ft/sec}$$

14. 45 mph = 66 ft/sec

30 mph = 44 ft/sec

$$a(t) = -a$$

$v(t) = -at + 66$ since $v(0) = 66$ ft/sec.

$s(t) = -\frac{a}{2}t^2 + 66t$ since $s(0) = 0$.

Solving the system

$$v(t) = -at + 66 = 44$$

$$s(t) = -\frac{a}{2}t^2 + 66t = 264$$

we obtain $t = 24/5$ and $a = 55/12$. We now solve $-(55/12)t + 66 = 0$ and get $t = 72/5$. Thus,

$$s\left(\frac{72}{5}\right) = -\frac{55/12}{2}\left(\frac{72}{5}\right)^2 + 66\left(\frac{72}{5}\right) \approx 475.2 \text{ ft.}$$

Stopping distance from 30 mph to rest is

$$475.2 - 264 = 211.2 \text{ ft.}$$

15. $a(t) = -32$

$$v(t) = -32t + 96$$

$$s(t) = -16t^2 + 96t$$

(a) $v(t) = -32t + 96 = 0$ when $t = 3$ sec.

(b) $s(3) = -144 + 288 = 144$ ft

(c) $v(t) = -32t + 96 = \frac{96}{2}$ when $t = \frac{3}{2}$ sec.

(d) $s\left(\frac{3}{2}\right) = -16\left(\frac{9}{4}\right) + 96\left(\frac{3}{2}\right) = 108$ ft

16. $a(t) = -9.8 \text{ m/sec}^2$

$$v(t) = -9.8t + v_0 = -9.8t + 40$$

$$s(t) = -4.9t^2 + 40t \quad (s(0) = 0)$$

(a) $v(t) = -9.8t + 40 = 0$ when $t = \frac{40}{9.8} \approx 4.08$ sec.

(b) $s(4.08) \approx 81.63$ m

(c) $v(t) = -9.8t + 40 = 20$ when $t = \frac{20}{9.8} \approx 2.04$ sec.

(d) $s(2.04) \approx 61.2$ m

17. $\displaystyle\sum_{i=1}^{8} \frac{1}{4i} = \frac{1}{4(1)} + \frac{1}{4(2)} + \cdots + \frac{1}{4(8)}$

18. $\displaystyle\sum_{i=1}^{12} \frac{i+2}{2i} = \frac{1+2}{2(1)} + \frac{2+2}{2(2)} + \frac{3+2}{2(3)} + \cdots + \frac{12+2}{2(12)}$

19. $\displaystyle\sum_{i=1}^{n} \left(\frac{3}{n}\right)\left(\frac{i+1}{n}\right)^2 = \frac{3}{n}\left(\frac{1+1}{n}\right)^2 + \frac{3}{n}\left(\frac{2+1}{n}\right)^2 + \cdots + \frac{3}{n}\left(\frac{n+1}{n}\right)^2$

20. $\displaystyle\sum_{i=1}^{n} 3i\left[2 + \frac{(i+1)^2}{n}\right] = 3\left[2 + \frac{4}{n}\right] + 6\left[2 + \frac{9}{n}\right] + \cdots + 3n\left[2 + \frac{(n+1)^2}{n}\right]$

21. $\displaystyle\sum_{i=1}^{10} 3i = 3\left(\frac{10(11)}{2}\right) = 165$

22. $\displaystyle\sum_{i=1}^{20} (4i - 1) = 4\frac{20(21)}{2} - 20 = 820$

23. $\displaystyle\sum_{i=1}^{20} (i+1)^2 = \sum_{i=1}^{20} (i^2 + 2i + 1)$

$$= \frac{20(21)(41)}{6} + 2\frac{20(21)}{2} + 20$$

$$= 2870 + 420 + 20 = 3310$$

24. $\displaystyle\sum_{i=1}^{12} i(i^2 - 1) = \sum_{i=1}^{12} (i^3 - i)$

$$= \frac{(12^2)(13^2)}{4} - \frac{12(13)}{2}$$

$$= 6084 - 78 = 6006$$

25. (a) $\sum_{i=1}^{10}(2i - 1)$

(b) $\sum_{i=1}^{n} i^3$

(c) $\sum_{i=1}^{10}(4i + 2)$

26. $x_1 = 2, x_2 = -1, x_3 = 5, x_4 = 3, x_5 = 7$

(a) $\frac{1}{5}\sum_{i=1}^{5} x_i = \frac{1}{5}(2 - 1 + 5 + 3 + 7) = \frac{16}{5}$

(b) $\sum_{i=1}^{5} \frac{1}{x_i} = \frac{1}{2} - 1 + \frac{1}{5} + \frac{1}{3} + \frac{1}{7} = \frac{37}{210}$

(c) $\sum_{i=1}^{5}(2x_i - x_i^2) = [2(2) - (2)^2] + [2(-1) - (-1)^2] + [2(5) - (5)^2] + [2(3) - (3)^2] + [2(7) - (7)^2] = -56$

(d) $\sum_{i=2}^{5}(x_i - x_{i-1}) = (-1 - 2) + [5 - (-1)] + (3 - 5) + (7 - 3) = 5$

27. $y = \frac{10}{x^2 + 1}, \Delta x = \frac{1}{2}, n = 4$

$$S(n) = S(4) = \frac{1}{2}\left[\frac{10}{1} + \frac{10}{(1/2)^2 + 1} + \frac{10}{(1)^2 + 1} + \frac{10}{(3/2)^2 + 1}\right]$$

$$\approx 13.0385$$

$$s(n) = s(4) = \frac{1}{2}\left[\frac{10}{(1/2)^2 + 1} + \frac{10}{1 + 1} + \frac{10}{(3/2)^2 + 1} + \frac{10}{2^2 + 1}\right]$$

$$\approx 9.0385$$

$9.0385 < \text{Area of Region} < 13.0385$

28. $y = 9 - \frac{1}{4}x^2, \Delta x = 1, n = 4$

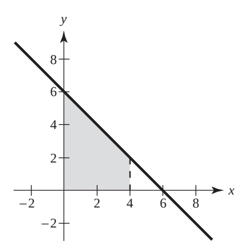
$$S(4) = 1\left[\left(9 - \frac{1}{4}(4)\right) + \left(9 - \frac{1}{4}(9)\right) + \left(9 - \frac{1}{4}(16)\right) + 9 - \frac{1}{4}(25)\right]$$

$$\approx 22.5$$

$$s(4) = 1\left[\left(9 - \frac{1}{4}(9)\right) + \left(9 - \frac{1}{4}(16)\right) + \left(9 - \frac{1}{4}(25)\right) + (9 - 9)\right]$$

$$\approx 14.5$$

29. $y = 6 - x, \Delta x = \frac{4}{n}$, right endpoints



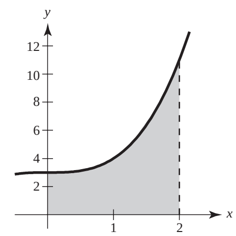
$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(6 - \frac{4i}{n}\right)\frac{4}{n}$$

$$= \lim_{n\to\infty}\frac{4}{n}\left[6n - \frac{4}{n}\frac{n(n + 1)}{2}\right]$$

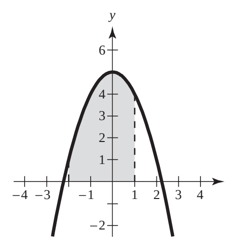
$$= \lim_{n\to\infty}\left[24 - 8\frac{n + 1}{n}\right] = 24 - 8 = 16$$

30. $y = x^2 + 3, \Delta x = \dfrac{2}{n}$ right endpoints



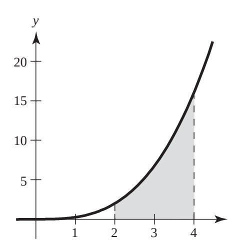
$$\begin{aligned}\text{Area} &= \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^2 + 3\right]\left(\frac{2}{n}\right)\\ &= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[\frac{4i^2}{n^2} + 3\right]\\ &= \lim_{n\to\infty}\frac{2}{n}\left[\frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} + 3n\right]\\ &= \lim_{n\to\infty}\left[\frac{4}{3}\frac{(n+1)(2n+1)}{n^2} + 6\right] = \frac{8}{3} + 6 = \frac{26}{3}\end{aligned}$$

31. $y = 5 - x^2, \Delta x = \dfrac{3}{n}$



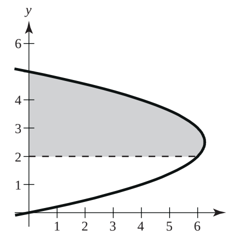
$$\begin{aligned}\text{Area} &= \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\left[5 - \left(-2 + \frac{3i}{n}\right)^2\right]\left(\frac{3}{n}\right)\\ &= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[1 + \frac{12i}{n} - \frac{9i^2}{n^2}\right]\\ &= \lim_{n\to\infty}\frac{3}{n}\left[n + \frac{12}{n}\frac{n(n+1)}{2} - \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right]\\ &= \lim_{n\to\infty}\left[3 + 18\frac{n+1}{n} - \frac{9}{2}\frac{(n+1)(2n+1)}{n^2}\right]\\ &= 3 + 18 - 9 = 12\end{aligned}$$

32. $y = \dfrac{1}{4}x^3, \Delta x = \dfrac{2}{n}$



$$\begin{aligned}\text{Area} &= \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{4}\left(2 + \frac{2i}{n}\right)^3\left(\frac{2}{n}\right)\\ &= \lim_{n\to\infty}\frac{1}{2n}\sum_{i=1}^{n}\left[8 + \frac{24i}{n} + \frac{24i^2}{n^2} + \frac{8i^3}{n^3}\right]\\ &= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left[1 + \frac{3i}{n} + \frac{3i^2}{n^2} + \frac{i^3}{n^3}\right]\\ &= \lim_{n\to\infty}\frac{4}{n}\left[n + \frac{3}{n}\frac{n(n+1)}{2} + \frac{3}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{1}{n^3}\frac{n^2(n+1)^2}{4}\right]\\ &= 4 + 6 + 4 + 1 = 15\end{aligned}$$

33. $x = 5y - y^2, 2 \le y \le 5, \Delta y = \dfrac{3}{n}$



$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \left[5\left(2 + \frac{3i}{n}\right) - \left(2 + \frac{3i}{n}\right)^2\right]\left(\frac{3}{n}\right)$$

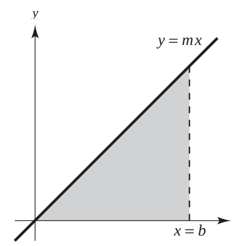
$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[10 + \frac{15i}{n} - 4 - 12\frac{i}{n} - \frac{9i^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[6 + \frac{3i}{n} - \frac{9i^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[6n + \frac{3}{n}\frac{n(n+1)}{2} - \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \left[18 + \frac{9}{2} - 9\right] = \frac{27}{2}$$

34. (a) $S = m\left(\frac{b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{2b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{3b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{4b}{4}\right)\left(\frac{b}{4}\right) = \frac{mb^2}{16}(1 + 2 + 3 + 4) = \frac{5mb^2}{8}$



$s = m(0)\left(\frac{b}{4}\right) + m\left(\frac{b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{2b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{3b}{4}\right)\left(\frac{b}{4}\right) = \frac{mb^2}{16}(1 + 2 + 3) = \frac{3mb^2}{8}$

(b) $S(n) = \sum_{i=1}^{n} f\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = \sum_{i=1}^{n}\left(\frac{mbi}{n}\right)\left(\frac{b}{n}\right) = m\left(\frac{b}{n}\right)^2 \sum_{i=1}^{n} i = \frac{mb^2}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{mb^2(n+1)}{2n}$

$s(n) = \sum_{i=0}^{n-1} f\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = \sum_{i=0}^{n-1} m\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = m\left(\frac{b}{n}\right)^2 \sum_{i=0}^{n-1} i = \frac{mb^2}{n^2}\left(\frac{(n-1)n}{2}\right) = \frac{mb^2(n-1)}{2n}$

(c) $\text{Area} = \lim_{n\to\infty} \frac{mb^2(n+1)}{2n} = \lim_{n\to\infty} \frac{mb^2(n-1)}{2n} = \frac{1}{2}mb^2 = \frac{1}{2}(b)(mb) = \frac{1}{2}(\text{base})(\text{height})$

(d) $\int_0^b mx\,dx = \left[\frac{1}{2}mx^2\right]_0^b = \frac{1}{2}mb^2$

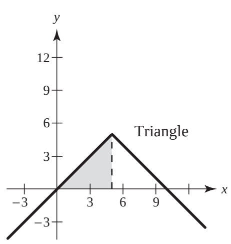
35. $\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (2c_i - 3)\,\Delta x_i = \int_4^6 (2x - 3)\,dx$

36. $\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 3c_i(9 - c_i^2)\,\Delta x_i = \int_1^3 3x(9 - x^2)\,dx$

37. $\int_{-2}^{0} (3x + 6)\,dx$

38. $\int_{-3}^{3} (9 - x^2)dx$

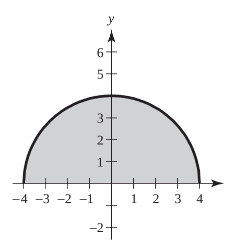
39.



$\int_0^5 (5 - |x - 5|)\,dx = \int_0^5 (5 - (5 - x))\,dx = \int_0^5 x\,dx = \frac{25}{2}$

(triangle)

40.



$\int_{-4}^{4} \sqrt{16 - x^2}\,dx = \frac{1}{2}\pi(4)^2 = 8\pi$

(semicircle)

41. (a) $\int_2^6 [f(x) + g(x)]\,dx = \int_2^6 f(x)\,dx + \int_2^6 g(x)\,dx = 10 + 3 = 13$

(b) $\int_2^6 [f(x) - g(x)]\,dx = \int_2^6 f(x)\,dx - \int_2^6 g(x)\,dx = 10 - 3 = 7$

(c) $\int_2^6 [2f(x) - 3g(x)]\,dx = 2\int_2^6 f(x)\,dx - 3\int_2^6 g(x)\,dx = 2(10) - 3(3) = 11$

(d) $\int_2^6 5f(x)\,dx = 5\int_2^6 f(x)\,dx = 5(10) = 50$

42. (a) $\int_0^6 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx = 4 + (-1) = 3$

(b) $\int_6^3 f(x)\,dx = -\int_3^6 f(x)\,dx = -(-1) = 1$

(c) $\int_4^4 f(x)\,dx = 0$

(d) $\int_3^6 -10\,f(x)\,dx = -10\int_3^6 f(x)\,dx = -10(-1) = 10$

43. $\int_0^4 (2 + x)\,dx = \left[2x + \frac{x^2}{2}\right]_0^4 = 8 + \frac{16}{2} = 16$

44. $\int_{-1}^1 (t^2 + 2)\,dt = \left[\frac{t^3}{3} + 2t\right]_{-1}^1 = \frac{14}{3}$

45. $\int_{-1}^1 (4t^3 - 2t)\,dt = \left[t^4 - t^2\right]_{-1}^1 = 0$

46. $\int_{-2}^2 (x^4 + 2x^2 - 5)\,dx = \left[\frac{x^5}{5} + \frac{2x^3}{3} - 5x\right]_{-2}^2$

$$= \left(\frac{32}{5} + \frac{16}{3} - 10\right) - \left(-\frac{32}{5} - \frac{16}{3} + 10\right)$$

$$= \frac{52}{15}$$

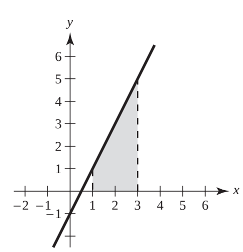
47. $\int_4^9 x\sqrt{x}\,dx = \int_4^9 x^{3/2}\,dx = \left[\frac{2}{5}x^{5/2}\right]_4^9 = \frac{2}{5}\left[\left(\sqrt{9}\right)^5 - \left(\sqrt{4}\right)^5\right] = \frac{2}{5}(243 - 32) = \frac{422}{5}$

48. $\int_1^2 \left(\frac{1}{x^2} - \frac{1}{x^3}\right)dx = \int_1^2 (x^{-2} - x^{-3})\,dx = \left[-\frac{1}{x} + \frac{1}{2x^2}\right]_1^2 = \left(-\frac{1}{2} + \frac{1}{8}\right) - \left(-1 + \frac{1}{2}\right) = \frac{1}{8}$

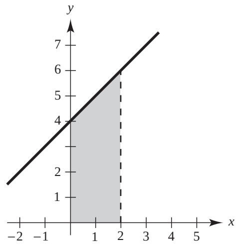
49. $\int_0^{3\pi/4} \sin\theta\,d\theta = \left[-\cos\theta\right]_0^{3\pi/4} = -\left(-\frac{\sqrt{2}}{2}\right) + 1 = 1 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2} + 2}{2}$

50. $\int_{-\pi/4}^{\pi/4} \sec^2 t\,dt = \left[\tan t\right]_{-\pi/4}^{\pi/4} = 1 - (-1) = 2$

51. $\int_1^3 (2x - 1)\,dx = \left[x^2 - x\right]_1^3 = 6$



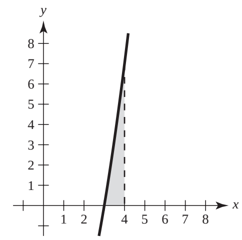
52. $\int_0^2 (x + 4)\,dx = \left[\frac{x^2}{2} + 4x\right]_0^2 = 10$



53. $\int_3^4 (x^2 - 9)\,dx = \left[\frac{x^3}{3} - 9x\right]_3^4$

$= \left(\frac{64}{3} - 36\right) - (9 - 27)$

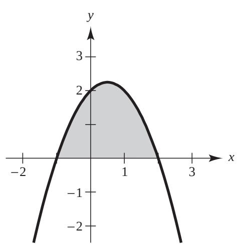
$= \frac{64}{3} - \frac{54}{3} = \frac{10}{3}$



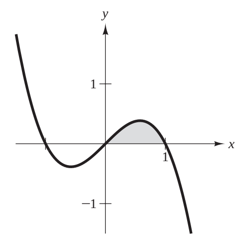
54. $\int_{-1}^2 (-x^2 + x + 2)\,dx = \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_{-1}^2$

$= \left(-\frac{8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$

$= \frac{10}{3} + \frac{7}{6} = \frac{9}{2}$



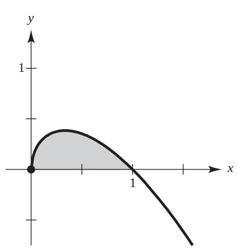
55. $\int_0^1 (x - x^3)\,dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$



56. $\int_0^1 \sqrt{x}(1 - x)\,dx = (x^{1/2} - x^{3/2})\,dx$

$= \left[\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^1$

$= \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$



57. Area $= \int_0^1 \sin x\,dx$

$= -\cos x\Big]_0^1$

$= -\cos(1) + 1$

$= 1 - \cos(1)$

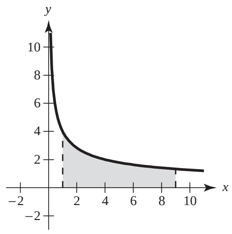
≈ 0.460

58. Area $= \int_0^{\pi/2} (x + \cos x)\,dx$

$= \left[\frac{x^2}{2} + \sin x\right]_0^{\pi/2}$

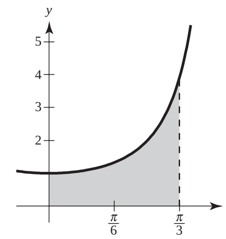
$= \frac{\pi^2}{8} + 1$

59. Area $= \int_1^9 \frac{4}{\sqrt{x}}\,dx = \left[\frac{4x^{1/2}}{(1/2)}\right]_1^9 = 8(3 - 1) = 16$



60. Area $= \int_0^{\pi/3} \sec^2 x\,dx$

$= \tan x\Big]_0^{\pi/3} = \sqrt{3}$

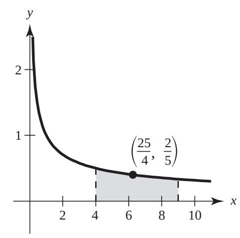


61. $\frac{1}{9-4}\int_4^9 \frac{1}{\sqrt{x}}\,dx = \left[\frac{1}{5}2\sqrt{x}\right]_4^9 = \frac{2}{5}(3-2) = \frac{2}{5}$ Average value

$\frac{2}{5} = \frac{1}{\sqrt{x}}$

$\sqrt{x} = \frac{5}{2}$

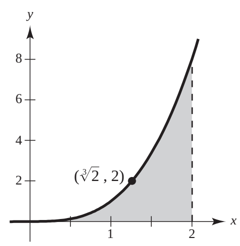
$x = \frac{25}{4}$



62. $\frac{1}{2-0}\int_0^2 x^3\,dx = \left[\frac{x^4}{8}\right]_0^2 = 2$

$x^3 = 2$

$x = \sqrt[3]{2}$



63. $F'(x) = x^2\sqrt{1 + x^3}$

64. $F'(x) = \frac{1}{x^2}$

65. $F'(x) = x^2 + 3x + 2$

66. $F'(x) = \csc^2 x$

67. $\int (x^2 + 1)^3\,dx = \int (x^6 + 3x^4 + 3x^2 + 1)\,dx = \frac{x^7}{7} + \frac{3}{5}x^5 + x^3 + x + C$

68. $\int \left(x + \frac{1}{x}\right)^2 dx = \int (x^2 + 2 + x^{-2})\,dx = \frac{x^3}{3} + 2x - \frac{1}{x} + C$

69. $u = x^3 + 3$, $du = 3x^2\,dx$

$\int \frac{x^2}{\sqrt{x^3 + 3}}\,dx = \int (x^3 + 3)^{-1/2}\,x^2\,dx = \frac{1}{3}\int (x^3 + 3)^{-1/2}\,3x^2\,dx = \frac{2}{3}(x^3 + 3)^{1/2} + C$

70. $u = x^3 + 3$, $du = 3x^2\,dx$

$\int x^2\sqrt{x^3 + 3}\,dx = \frac{1}{3}\int (x^3 + 3)^{1/2}\,3x^2\,dx = \frac{2}{9}(x^3 + 3)^{3/2} + C$

71. $u = 1 - 3x^2$, $du = -6x\,dx$

$\int x(1 - 3x^2)^4\,dx = -\frac{1}{6}\int (1 - 3x^2)^4(-6x\,dx) = -\frac{1}{30}(1 - 3x^2)^5 + C = \frac{1}{30}(3x^2 - 1)^5 + C$

72. $u = x^2 + 6x - 5$, $du = (2x + 6)\,dx$

$$\int \frac{x+3}{(x^2+6x-5)^2} = \frac{1}{2}\int \frac{2x+6}{(x^2+6x-5)^2}\,dx = \frac{-1}{2}(x^2+6x-5)^{-1} + C = \frac{-1}{2(x^2+6x-5)} + C$$

73. $\displaystyle\int \sin^3 x \cos x\,dx = \frac{1}{4}\sin^4 x + C$

74. $\displaystyle\int x \sin 3x^2\,dx = \frac{1}{6}\int (\sin 3x^2)(6x)\,dx = -\frac{1}{6}\cos 3x^2 + C$

75. $\displaystyle\int \frac{\sin\theta}{\sqrt{1-\cos\theta}}\,d\theta = \int (1-\cos\theta)^{-1/2}\sin\theta\,d\theta = 2(1-\cos\theta)^{1/2} + C = 2\sqrt{1-\cos\theta} + C$

76. $\displaystyle\int \frac{\cos x}{\sqrt{\sin x}}\,dx = \int (\sin x)^{-1/2}\cos x\,dx = 2(\sin x)^{1/2} + C = 2\sqrt{\sin x} + C$

77. $\displaystyle\int \tan^n x \sec^2 x\,dx = \frac{\tan^{n+1} x}{n+1} + C,\ n \neq -1$

78. $\displaystyle\int \sec 2x \tan 2x\,dx = \frac{1}{2}\int (\sec 2x \tan 2x)(2)\,dx = \frac{1}{2}\sec 2x + C$

79. $\displaystyle\int (1+\sec \pi x)^2 \sec \pi x \tan \pi x\,dx = \frac{1}{\pi}\int (1+\sec \pi x)^2(\pi \sec \pi x \tan \pi x)\,dx = \frac{1}{3\pi}(1+\sec \pi x)^3 + C$

80. $\displaystyle\int \cot^4 \alpha \csc^2 \alpha\,d\alpha = -\int (\cot\alpha)^4(-\csc^2\alpha)\,d\alpha = -\frac{1}{5}\cot^5\alpha + C$

81. $\displaystyle\int_{-1}^{2} x(x^2-4)\,dx = \frac{1}{2}\int_{-1}^{2}(x^2-4)(2x)\,dx = \frac{1}{2}\frac{(x^2-4)^2}{2}\Bigg]_{-1}^{2} = \frac{1}{4}[0-9] = -\frac{9}{4}$

82. $\displaystyle\int_0^1 x^2(x^3+1)^3\,dx = \frac{1}{3}\int_0^1 (x^3+1)^3(3x^2)\,dx = \frac{1}{12}\Big[(x^3+1)^4\Big]_0^1 = \frac{1}{12}(16-1) = \frac{5}{4}$

83. $\displaystyle\int_0^3 \frac{1}{\sqrt{1+x}}\,dx = \int_0^3 (1+x)^{-1/2}\,dx = \Big[2(1+x)^{1/2}\Big]_0^3 = 4 - 2 = 2$

84. $\displaystyle\int_3^6 \frac{x}{3\sqrt{x^2-8}}\,dx = \frac{1}{6}\int_3^6 (x^2-8)^{-1/2}(2x)\,dx = \left[\frac{1}{3}(x^2-8)^{1/2}\right]_3^6 = \frac{1}{3}\left(2\sqrt{7}-1\right)$

85. $u = 1 - y,\ y = 1 - u,\ dy = -du$

When $y = 0$, $u = 1$. When $y = 1$, $u = 0$.

$$2\pi\int_0^1 (y+1)\sqrt{1-y}\,dy = 2\pi\int_1^0 -[(1-u)+1]\sqrt{u}\,du$$

$$= 2\pi\int_1^0 (u^{3/2} - 2u^{1/2})\,du = 2\pi\left[\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2}\right]_1^0 = \frac{28\pi}{15}$$

86. $u = x + 1,\ x = u - 1,\ dx = du$

When $x = -1$, $u = 0$. When $x = 0$, $u = 1$.

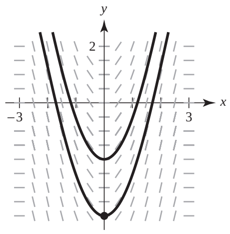
$$2\pi\int_{-1}^0 x^2\sqrt{x+1}\,dx = 2\pi\int_0^1 (u-1)^2\sqrt{u}\,du$$

$$= 2\pi\int_0^1 (u^{5/2} - 2u^{3/2} + u^{1/2})\,du = 2\pi\left[\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right]_0^1 = \frac{32\pi}{105}$$

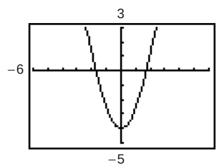
87. $\int_0^{\pi} \cos\left(\frac{x}{2}\right) dx = 2\int_0^{\pi} \cos\left(\frac{x}{2}\right)\frac{1}{2}\, dx = \left[2\sin\left(\frac{x}{2}\right)\right]_0^{\pi} = 2$

88. $\int_{-\pi/4}^{\pi/4} \sin 2x\, dx = 0$ since $\sin 2x$ is an odd function.

89. (a)



(b) $\frac{dy}{dx} = x\sqrt{9 - x^2}, \quad (0, -4)$

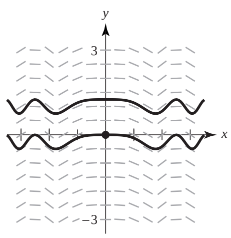


$$y = \int (9 - x^2)^{1/2} x\, dx = \frac{-1}{2}\frac{(9 - x^2)^{3/2}}{3/2} + C = -\frac{1}{3}(9 - x^2)^{3/2} + C$$

$$-4 = -\frac{1}{3}(9 - 0)^{3/2} + C = -\frac{1}{3}(27) + C \implies C = 5$$

$$y = -\frac{1}{3}(9 - x^2)^{3/2} + 5$$

90. (a)



(b) $\frac{dy}{dx} = -\frac{1}{2}x\sin(x^2), \quad (0, 0)$

$$y = \int -\frac{1}{2}x\sin(x^2)\, dx = -\frac{1}{4}\int \sin(x^2)(2x\, dx) \qquad (u = x^2)$$

$$= -\frac{1}{4}(-\cos(x^2)) + C$$

$$= \frac{1}{4}\cos(x^2) + C$$

$$0 = \frac{1}{4}\cos(0) + C \implies C = -\frac{1}{4}$$

$$y = \frac{1}{4}\cos(x^2) - \frac{1}{4}$$

91. $\int_1^9 x(x - 1)^{1/3}\, dx$. Let $u = x - 1$, $du = dx$.

$$A = \int_0^8 (u + 1)u^{1/3}\, du = \int_0^8 (u^{4/3} + u^{1/3})\, du$$

$$= \left[\frac{3u^{7/3}}{7} + \frac{3u^{4/3}}{4}\right]_0^8$$

$$= \frac{3}{7}(128) + \frac{3}{4}(16) = \frac{468}{7}$$

92. $\int_0^{\pi/2} (\cos x + \sin(2x))\, dx = \left[\sin x - \frac{1}{2}\cos(2x)\right]_0^{\pi/2}$

$$= \left(1 + \frac{1}{2}\right) - \left(0 - \frac{1}{2}\right) = 2$$

93. $p = 1.20 + 0.04t$

$$C = \frac{15{,}000}{M}\int_t^{t+1} p\, ds = \frac{15{,}000}{M}\int_t^{t+1} (1.20 + 0.04s)\, ds$$

(a) 2000 corresponds to $t = 10$.

$$C = \frac{15{,}000}{M}\int_{10}^{11} [1.20 + 0.04t]\, dt$$

$$= \frac{15{,}000}{M}\left[1.20t + 0.02t^2\right]_{10}^{11} = \frac{24{,}300}{M}$$

(b) 2005 corresponds to $t = 15$.

$$C = \frac{15{,}000}{M}\left[1.20t + 0.02t^2\right]_{15}^{16} = \frac{27{,}300}{M}$$

94. $\int_0^2 1.75 \sin \frac{\pi t}{2}\, dt = -\frac{2}{\pi}\left[1.75 \cos \frac{\pi t}{2}\right]_0^2 = -\frac{2}{\pi}(1.75)(-1 - 1) = \frac{7}{\pi} \approx 2.2282$ liters

Increase is

$\frac{7}{\pi} - \frac{5.1}{\pi} = \frac{1.9}{\pi} \approx 0.6048$ liters.

95. Trapezoidal Rule $(n = 4)$: $\int_1^2 \frac{1}{1 + x^3}\, dx \approx \frac{1}{8}\left[\frac{1}{1 + 1^3} + \frac{2}{1 + (1.25)^3} + \frac{2}{1 + (1.5)^3} + \frac{2}{1 + (1.75)^3} + \frac{1}{1 + 2^3}\right] \approx 0.257$

Simpson's Rule $(n = 4)$: $\int_1^2 \frac{1}{1 + x^3}\, dx \approx \frac{1}{12}\left[\frac{1}{1 + 1^3} + \frac{4}{1 + (1.25)^3} + \frac{2}{1 + (1.5)^3} + \frac{4}{1 + (1.75)^3} + \frac{1}{1 + 2^3}\right] \approx 0.254$

Graphing utility: 0.254

96. Trapezoidal Rule $(n = 4)$: $\int_0^1 \frac{x^{3/2}}{3 - x^2}\, dx \approx \frac{1}{8}\left[0 + \frac{2(1/4)^{3/2}}{3 - (1/4)^2} + \frac{2(1/2)^{3/2}}{3 - (1/2)^2} + \frac{2(3/4)^{3/2}}{3 - (3/4)^2} + \frac{1}{2}\right] \approx 0.172$

Simpson's Rule $(n = 4)$: $\int_0^1 \frac{x^{3/2}}{3 - x^2}\, dx \approx \frac{1}{12}\left[0 + \frac{4(1/4)^{3/2}}{3 - (1/4)^2} + \frac{2(1/2)^{3/2}}{3 - (1/2)^2} + \frac{4(3/4)^{3/2}}{3 - (3/4)^2} + \frac{1}{2}\right] \approx 0.166$

Graphing utility: 0.166

97. Trapezoidal Rule $(n = 4)$: $\int_0^{\pi/2} \sqrt{x} \cos x\, dx \approx 0.637$

Simpson's Rule $(n = 4)$: 0.685

Graphing Utility: 0.704

98. Trapezoidal Rule $(n = 4)$: $\int_0^{\pi} \sqrt{1 + \sin^2 x}\, dx \approx 3.820$

Simpson's Rule $(n = 4)$: 3.820

Graphing utility: 3.820

Problem Solving for Chapter 4

1. (a) $L(1) = \int_1^1 \frac{1}{t}\, dt = 0$

(b) $L'(x) = \frac{1}{x}$ by the Second Fundamental Theorem of Calculus.

$L'(1) = 1$

(c) $L(x) = 1 = \int_1^x \frac{1}{t}\, dt$ for $x \approx 2.718$

$\int_1^{2.718} \frac{1}{t}\, dt = 0.999896$

(**Note:** The exact value of x is e, the base of the natural logarithm function.)

(d) We first show that $\int_1^{x_1} \frac{1}{t}\, dt = \int_{1/x_1}^1 \frac{1}{t}\, dt$.

To see this, let $u = \frac{t}{x_1}$ and $du = \frac{1}{x_1}\, dt$.

Then $\int_1^{x_1} \frac{1}{t}\, dt = \int_{1/x_1}^1 \frac{1}{ux_1}(x_1\, du) = \int_{1/x_1}^1 \frac{1}{u}\, du = \int_{1/x_1}^1 \frac{1}{t}\, dt$.

Now, $L(x_1 x_2) = \int_1^{x_1 x_2} \frac{1}{t}\, dt = \int_{1/x_1}^{x_2} \frac{1}{u}\, du\left(\text{using } u = \frac{t}{x_1}\right)$

$= \int_{1/x_1}^1 \frac{1}{u}\, du + \int_1^{x_2} \frac{1}{u}\, du$

$= \int_1^{x_1} \frac{1}{u}\, du + \int_1^{x_2} \frac{1}{u}\, du$

$= L(x_1) + L(x_2)$.

2. (a) $F(x) = \int_2^x \sin t^2\, dt$

x	0	1.0	1.5	1.9	2.0	2.1	2.5	3.0	4.0	5.0
$F(x)$	−0.8048	−0.4945	−0.0265	0.0611	0	−0.0867	−0.3743	−0.0312	−0.0576	−0.2769

(b) $G(x) = \dfrac{1}{x-2}\int_2^x \sin t^2\, dt$

x	1.9	1.95	1.99	2.01	2.1
$G(x)$	−0.6106	−0.6873	−0.7436	−0.7697	−0.8671

$\lim_{x\to 2} G(x) \approx -0.75$

(c) $$F'(2) = \lim_{x\to 2}\frac{F(x)-F(2)}{x-2} = \lim_{x\to 2}\frac{1}{x-2}\int_2^x \sin t^2\, dt = \lim_{x\to 2} G(x)$$

Since $F'(x) = \sin x^2$, $F'(2) = \sin 4 = \lim_{x\to 2} G(x)$. (**Note:** $\sin 4 \approx -0.7568$)

3. $y = x^4 - 4x^3 + 4x^2$, $[0, 2]$, $c_i = \dfrac{2i}{n}$

(a) $\Delta x = \dfrac{2}{n}$, $f(x) = x^4 - 4x^3 + 4x^2$

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^4 - 4\left(\frac{2i}{n}\right)^3 + 4\left(\frac{2i}{n}\right)^2\right]\frac{2}{n}$$

(b) $$\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^4 - 4\left(\frac{2i}{n}\right)^3 + 4\left(\frac{2i}{n}\right)^2\right]\frac{2}{n} = \left[\frac{8(n+1)(6n^3+9n^2+n-1)}{15n^3} - \frac{8(n+1)^2}{n} + \frac{8(n+1)(2n+1)}{3n}\right]\frac{2}{n} = \left[\frac{8(n+1)(n^3-n^2+n-1)}{15n^3}\right]\frac{2}{n}$$

(c) $$A = \lim_{n\to\infty}\left[\frac{8(n+1)(n^3-n^2+n-1)}{15n^3}\right]\frac{2}{n} = \frac{16}{15}$$

4. $y = \dfrac{1}{2}x^5 + 2x^3$, $[0, 2]$, $c_i = \dfrac{2i}{n}$, $\Delta x = \dfrac{2}{n}$

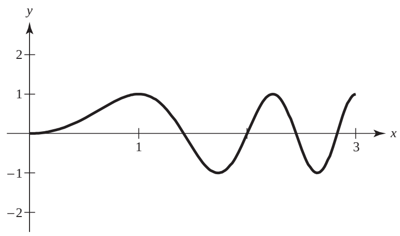
(a) $$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{1}{2}\left(\frac{2i}{n}\right)^5 + 2\left(\frac{2i}{n}\right)^3\right]\frac{2}{n}$$

(b) $$\sum_{i=1}^{n}\left[\frac{1}{2}\left(\frac{2i}{n}\right)^5 + 2\left(\frac{2i}{n}\right)^3\right]\frac{2}{n} = \left[\frac{4(n+1)^2(2n^2+2n-1)}{3n^3} + \frac{4(n+1)^2}{n}\right]\frac{2}{n} = \left[\frac{8(n+1)^2(5n^2+2n-1)}{3n^4}\right]$$

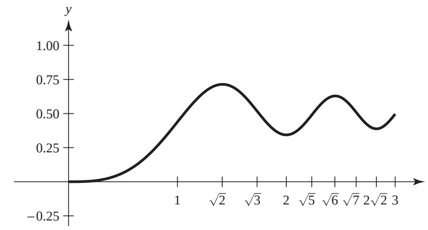
(c) $$A = \lim_{n\to\infty}\left[\frac{8(n+1)^2(5n^2+2n-1)}{3n^4}\right] = \frac{40}{3}$$

5. $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$

(a)



(b)



The zeros of $y = \sin\frac{\pi x^2}{2}$ correspond to the relative extrema of $S(x)$.

(c) $S'(x) = \sin\frac{\pi x^2}{2} = 0 \implies \frac{\pi x^2}{2} = n\pi \implies x^2 = 2n \implies x = \sqrt{2n}$, n integer.

Relative maximum at $x = \sqrt{2} \approx 1.4142$ and $x = \sqrt{6} \approx 2.4495$

Relative minimum at $x = 2$ and $x = \sqrt{8} \approx 2.8284$

(d) $S''(x) = \cos\left(\frac{\pi x^2}{2}\right)(\pi x) = 0 \implies \frac{\pi x^2}{2} = \frac{\pi}{2} + n\pi \implies x^2 = 1 + 2n \implies x = \sqrt{1 + 2n}$, n integer

Points of inflection at $x = 1, \sqrt{3}, \sqrt{5}$, and $\sqrt{7}$.

6. (a) $\int_{-1}^{1} \cos x\, dx \approx \cos\left(-\frac{1}{\sqrt{3}}\right) + \cos\left(\frac{1}{\sqrt{3}}\right) = 2\cos\left(\frac{1}{\sqrt{3}}\right) \approx 1.6758$

$\int_{-1}^{1} \cos x\, dx = \sin x\Big]_{-1}^{1} = 2\sin(1) \approx 1.6829$

Error: $|1.6829 - 1.6758| = 0.0071$

(b) $\int_{-1}^{1} \frac{1}{1 + x^2}\, dx \approx \frac{1}{1 + (1/3)} + \frac{1}{1 + (1/3)} = \frac{3}{2}$

(**Note:** exact answer is $\pi/2 \approx 1.5708$)

(c) Let $p(x) = ax^3 + bx^2 + cx + d$.

$\int_{-1}^{1} p(x)\, dx = \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_{-1}^{1} = \frac{2b}{3} + 2d$

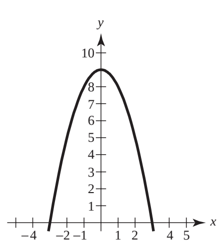
$p\left(-\frac{1}{\sqrt{3}}\right) + p\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{b}{3} + d\right) + \left(\frac{b}{3} + d\right) = \frac{2b}{3} + 2d$

7. (a) Area $= \int_{-3}^{3} (9 - x^2)\, dx = 2\int_0^3 (9 - x^2)\, dx$

$= 2\left[9x - \frac{x^3}{3}\right]_0^3$

$= 2[27 - 9] = 36$

(b) Base $= 6$, height $= 9$, Area $= \frac{2}{3}bh = \frac{2}{3}(6)(9) = 36$



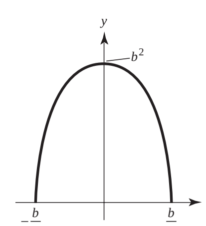
(c) Let the parabola be given by $y = b^2 - a^2x^2,\ a, b > 0$.

Area $= 2\int_0^{b/a} (b^2 - a^2x^2)\, dx$

$= 2\left[b^2x - a^2\frac{x^3}{3}\right]_0^{b/a}$

$= 2\left[b^2\left(\frac{b}{a}\right) - \frac{a^2}{3}\left(\frac{b}{a}\right)^3\right]$

$= 2\left[\frac{b^3}{a} - \frac{1}{3}\frac{b^3}{a}\right] = \frac{4}{3}\frac{b^3}{a}$



Base $= \frac{2b}{a}$, height $= b^2$

Archimedes' Formula: Area $= \frac{2}{3}\left(\frac{2b}{a}\right)(b^2) = \frac{4}{3}\frac{b^3}{a}$

8. Let d be the distance traversed and a be the uniform acceleration. We can assume that $v(0) = 0$ and $s(0) = 0$. Then

$$a(t) = a$$

$$v(t) = at$$

$$s(t) = \frac{1}{2}at^2.$$

$s(t) = d$ when $t = \sqrt{\frac{2d}{a}}$.

The highest speed is $v = a\sqrt{\frac{2d}{a}} = \sqrt{2ad}$.

The lowest speed is $v = 0$.

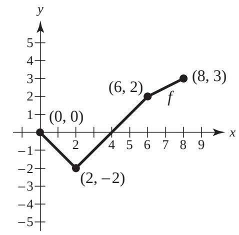
The mean speed is $\frac{1}{2}\left(\sqrt{2ad} + 0\right) = \sqrt{\frac{ad}{2}}$.

The time necessary to traverse the distance d at the mean speed is

$$t = \frac{d}{\sqrt{ad/2}} = \sqrt{\frac{2d}{a}}$$

which is the same as the time calculated above.

9. (a)



(b)

x	0	1	2	3	4	5	6	7	8
$F(x)$	0	$-\frac{1}{2}$	-2	$-\frac{7}{2}$	-4	$-\frac{7}{2}$	-2	$\frac{1}{4}$	3

(c) $f(x) = \begin{cases} -x, & 0 \le x < 2 \\ x - 4, & 2 \le x < 6 \\ \frac{1}{2}x - 1, & 6 \le x \le 8 \end{cases}$

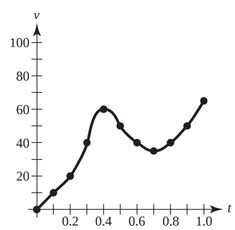
$$F(x) = \int_0^x f(t)\,dt = \begin{cases} (-x^2/2), & 0 \le x < 2 \\ (x^2/2) - 4x + 4, & 2 \le x < 6 \\ (1/4)x^2 - x - 5, & 6 \le x \le 8 \end{cases}$$

$F'(x) = f(x)$. F is decreasing on $(0, 4)$ and increasing on $(4, 8)$. Therefore, the minimum is -4 at $x = 4$, and the maximum is 3 at $x = 8$.

(d) $F''(x) = f'(x) = \begin{cases} -1, & 0 < x < 2 \\ 1, & 2 < x < 6 \\ \frac{1}{2}, & 6 < x < 8 \end{cases}$

$x = 2$ is a point of inflection, whereas $x = 6$ is not.

10. (a)



(b) v is increasing (positive acceleration) on $(0, 0.4)$ and $(0.7, 1.0)$.

(c) Average acceleration $= \frac{v(0.4) - v(0)}{0.4 - 0} = \frac{60 - 0}{0.4} = 150 \text{ mi/hr}^2$

(d) This integral is the total distance traveled in miles.

$$\int_0^1 v(t)\,dt \approx \frac{1}{10}[0 + 2(20) + 2(60) + 2(40) + 2(40) + 65] = \frac{385}{10} = 38.5 \text{ miles}$$

(e) One approximation is

$$a(0.8) \approx \frac{v(0.9) - v(0.8)}{0.9 - 0.8} = \frac{50 - 40}{0.1} = 100 \text{ mi/hr}^2$$

(other answers possible)

11. $\int_0^x f(t)(x - t)\,dt = \int_0^x x f(t)\,dt - \int_0^x t f(t)\,dt = x\int_0^x f(t)\,dt - \int_0^x t f(t)\,dt$

Thus, $\dfrac{d}{dx}\int_0^x f(t)(x - t)\,dt = x f(x) + \int_0^x f(t)\,dt - x f(x) = \int_0^x f(t)\,dt$

Differentiating the other integral,

$$\frac{d}{dx}\int_0^x \left(\int_0^x f(v)\,dv\right) dt = \int_0^x f(v)\,dv.$$

Thus, the two original integrals have equal derivatives,

$$\int_0^x f(t)(x - t)\,dt = \int_0^x \left(\int_0^t f(v)\,dv\right) dt + C.$$

Letting $x = 0$, we see that $C = 0$.

12. Consider $F(x) = [f(x)]^2 \implies F'(x) = 2f(x)\,f'(x)$. Thus,

$$\int_a^b f(x)\,f'(x)\,dx = \int_a^b \frac{1}{2}F'(x)\,dx$$
$$= \left[\frac{1}{2}F(x)\right]_a^b$$
$$= \frac{1}{2}[F(b) - F(a)]$$
$$= \frac{1}{2}[f(b)^2 - f(a)^2].$$

13. Consider $\int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big]_0^1 = \frac{2}{3}$. The corresponding Riemann Sum using right-hand endpoints is

$$S(n) = \frac{1}{n}\left[\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\right]$$
$$= \frac{1}{n^{3/2}}\left[\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}\right].$$

Thus, $\lim_{n\to\infty} \dfrac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n^{3/2}} = \dfrac{2}{3}$.

14. Consider $\int_0^1 x^5\,dx = \frac{x^6}{6}\Big]_0^1 = \frac{1}{6}$.

The corresponding Riemann Sum using right endpoints is

$$S(n) = \frac{1}{n}\left[\left(\frac{1}{n}\right)^5 + \left(\frac{2}{n}\right)^5 + \cdots + \left(\frac{n}{n}\right)^5\right]$$
$$= \frac{1}{n^6}[1^5 + 2^5 + \cdots + n^5].$$

Thus, $\lim_{n\to\infty} S(n) = \lim_{n\to\infty} \dfrac{1^5 + 2^5 + \cdots + n^5}{n^6} = \dfrac{1}{6}$.

15. By Theorem 4.8, $0 < f(x) \le M \implies \int_a^b f(x)\,dx \le \int_a^b M\,dx = M(b - a)$.

Similarly, $m \le f(x) \implies m(b - a) = \int_a^b m\,dx \le \int_a^b f(x)\,dx$.

Thus, $m(b - a) \le \int_a^b f(x)\,dx \le M(b - a)$. On the interval $[0, 1]$, $1 \le \sqrt{1 + x^4} \le \sqrt{2}$ and $b - a = 1$.

Thus, $1 \le \int_0^1 \sqrt{1 + x^4}\,dx \le \sqrt{2}$. $\left(\textbf{Note: } \int_0^1 \sqrt{1 + x^4}\,dx \approx 1.0894\right)$

16. (a) Let $A = \int_0^b \frac{f(x)}{f(x) + f(b - x)}\,dx.$

Let $u = b - x$, $du = -dx$.

$$A = \int_b^0 \frac{f(b-u)}{f(b-u) + f(u)}(-du)$$
$$= \int_0^b \frac{f(b-u)}{f(b-u) + f(u)}\,du$$
$$= \int_0^b \frac{f(b-x)}{f(b-x) + f(x)}\,dx$$

Then,

$$2A = \int_0^b \frac{f(x)}{f(x) + f(b-x)}\,dx + \int_0^b \frac{f(b-x)}{f(b-x) + f(x)}\,dx$$
$$= \int_0^b 1\,dx = b.$$

Thus, $A = \frac{b}{2}$.

(b) $b = 1 \implies \int_0^1 \frac{\sin x}{\sin(1-x) + \sin x}\,dx = \frac{1}{2}$

(c) $b = 3, f(x) = \sqrt{x}$

$$\int_0^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{3-x}}\,dx = \frac{3}{2}$$

17. (a) $(1 + i)^3 = 1 + 3i + 3i^2 + i^3 \implies (1 + i)^3 - i^3 = 3i^2 + 3i + 1$

(b) $3i^2 + 3i + 1 = (i + 1)^3 - i^3$

$$\sum_{i=1}^{n}(3i^2 + 3i + 1) = \sum_{i=1}^{n}[(i + 1)^3 - i^3]$$
$$= (2^3 - 1^3) + (3^3 - 2^3) + \cdots + [((n + 1)^3 - n^3)]$$
$$= (n + 1)^3 - 1$$

Hence, $(n + 1)^3 = \sum_{i=1}^{n}(3i^2 + 3i + 1) + 1$.

(c) $(n + 1)^3 - 1 = \sum_{i=1}^{n}(3i^2 + 3i + 1) = \sum_{i=1}^{n}3i^2 + \frac{3(n)(n + 1)}{2} + n$

$$\implies \sum_{i=1}^{n}3i^2 = n^3 + 3n^2 + 3n - \frac{3n(n + 1)}{2} - n$$
$$= \frac{2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n}{2}$$
$$= \frac{2n^3 + 3n^2 + n}{2}$$
$$= \frac{n(n + 1)(2n + 1)}{2}$$
$$\implies \sum_{i=1}^{n}i^2 = \frac{n(n + 1)(2n + 1)}{6}$$

18. Since $-|f(x)| \le f(x) \le |f(x)|$,

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx \implies \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

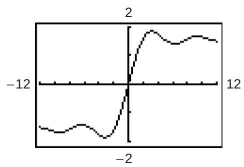
19. (a) $R < I < T < L$

(b) $S(4) = \frac{4 - 0}{3(4)}[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)]$

$$\approx \frac{1}{3}\left[4 + 4(2) + 2(1) + 4\left(\frac{1}{2}\right) + \frac{1}{4}\right] \approx 5.417$$

20. $\text{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt$

(a)



(b) $S'_i(x) = \frac{\sin x}{x}$ $\quad \text{Si}'(x) = 0$ for $x = 2n\pi$

For positive x, $x = (2n - 1)\pi$

For negative x, $x = 2n\pi$

Maxima at $\pi, 3\pi, 5\pi, \ldots$ and $-2\pi, -4\pi, -6\pi, \ldots$

(c) $\text{Si}''(x) = \frac{x \cos x - \sin x}{x^2} = 0$

$x \cos x = \sin x$ for $x \approx 4.4934$

$\text{Si}(4.4934) \approx 1.6556$

(d) Horizontal asymptotes at $y = \pm\frac{\pi}{2}$

$\lim_{x \to \infty} \text{Si}(x) = \frac{\pi}{2}$

$\lim_{x \to -\infty} \text{Si}(x) = \frac{-\pi}{2}$

94. $\int_0^2 1.75 \sin \frac{\pi t}{2}\, dt = -\frac{2}{\pi}\left[1.75 \cos \frac{\pi t}{2}\right]_0^2 = -\frac{2}{\pi}(1.75)(-1 - 1) = \frac{7}{\pi} \approx 2.2282$ liters

Increase is

$$\frac{7}{\pi} - \frac{5.1}{\pi} = \frac{1.9}{\pi} \approx 0.6048 \text{ liters.}$$

95. Trapezoidal Rule $(n = 4)$: $\int_1^2 \frac{1}{1 + x^3}\, dx \approx \frac{1}{8}\left[\frac{1}{1 + 1^3} + \frac{2}{1 + (1.25)^3} + \frac{2}{1 + (1.5)^3} + \frac{2}{1 + (1.75)^3} + \frac{1}{1 + 2^3}\right] \approx 0.257$

Simpson's Rule $(n = 4)$: $\int_1^2 \frac{1}{1 + x^3}\, dx \approx \frac{1}{12}\left[\frac{1}{1 + 1^3} + \frac{4}{1 + (1.25)^3} + \frac{2}{1 + (1.5)^3} + \frac{4}{1 + (1.75)^3} + \frac{1}{1 + 2^3}\right] \approx 0.254$

Graphing utility: 0.254

96. Trapezoidal Rule $(n = 4)$: $\int_0^1 \frac{x^{3/2}}{3 - x^2}\, dx \approx \frac{1}{8}\left[0 + \frac{2(1/4)^{3/2}}{3 - (1/4)^2} + \frac{2(1/2)^{3/2}}{3 - (1/2)^2} + \frac{2(3/4)^{3/2}}{3 - (3/4)^2} + \frac{1}{2}\right] \approx 0.172$

Simpson's Rule $(n = 4)$: $\int_0^1 \frac{x^{3/2}}{3 - x^2}\, dx \approx \frac{1}{12}\left[0 + \frac{4(1/4)^{3/2}}{3 - (1/4)^2} + \frac{2(1/2)^{3/2}}{3 - (1/2)^2} + \frac{4(3/4)^{3/2}}{3 - (3/4)^2} + \frac{1}{2}\right] \approx 0.166$

Graphing utility: 0.166

97. Trapezoidal Rule $(n = 4)$: $\int_0^{\pi/2} \sqrt{x} \cos x\, dx \approx 0.637$

Simpson's Rule $(n = 4)$: 0.685

Graphing Utility: 0.704

98. Trapezoidal Rule $(n = 4)$: $\int_0^{\pi} \sqrt{1 + \sin^2 x}\, dx \approx 3.820$

Simpson's Rule $(n = 4)$: 3.820

Graphing utility: 3.820

Problem Solving for Chapter 4

1. (a) $L(1) = \int_1^1 \frac{1}{t}\, dt = 0$

(b) $L'(x) = \frac{1}{x}$ by the Second Fundamental Theorem of Calculus.

$L'(1) = 1$

(c) $L(x) = 1 = \int_1^x \frac{1}{t}\, dt$ for $x \approx 2.718$

$$\int_1^{2.718} \frac{1}{t}\, dt = 0.999896$$

(**Note:** The exact value of x is e, the base of the natural logarithm function.)

(d) We first show that $\int_1^{x_1} \frac{1}{t}\, dt = \int_{1/x_1}^{1} \frac{1}{t}\, dt.$

To see this, let $u = \frac{t}{x_1}$ and $du = \frac{1}{x_1}\, dt.$

Then $\int_1^{x_1} \frac{1}{t}\, dt = \int_{1/x_1}^{1} \frac{1}{ux_1}(x_1\, du) = \int_{1/x_1}^{1} \frac{1}{u}\, du = \int_{1/x_1}^{1} \frac{1}{t}\, dt.$

$$\begin{aligned}
\text{Now, } L(x_1x_2) &= \int_1^{x_1x_2} \frac{1}{t}\, dt = \int_{1/x_1}^{x_2} \frac{1}{u}\, du \left(\text{using } u = \frac{t}{x_1}\right) \\
&= \int_{1/x_1}^{1} \frac{1}{u}\, du + \int_1^{x_2} \frac{1}{u}\, du \\
&= \int_1^{x_1} \frac{1}{u}\, du + \int_1^{x_2} \frac{1}{u}\, du \\
&= L(x_1) + L(x_2).
\end{aligned}$$

2. (a) $F(x) = \int_2^x \sin t^2\, dt$

x	0	1.0	1.5	1.9	2.0	2.1	2.5	3.0	4.0	5.0
$F(x)$	−0.8048	−0.4945	−0.0265	0.0611	0	−0.0867	−0.3743	−0.0312	−0.0576	−0.2769

(b) $G(x) = \dfrac{1}{x-2}\int_2^x \sin t^2\, dt$

x	1.9	1.95	1.99	2.01	2.1
$G(x)$	−0.6106	−0.6873	−0.7436	−0.7697	−0.8671

$\lim_{x\to 2} G(x) \approx -0.75$

(c) $$F'(2) = \lim_{x\to 2}\frac{F(x) - F(2)}{x-2} = \lim_{x\to 2}\frac{1}{x-2}\int_2^x \sin t^2\, dt = \lim_{x\to 2} G(x)$$

Since $F'(x) = \sin x^2$, $F'(2) = \sin 4 = \lim_{x\to 2} G(x)$. (**Note:** $\sin 4 \approx -0.7568$)

3. $y = x^4 - 4x^3 + 4x^2$, $[0, 2]$, $c_i = \dfrac{2i}{n}$

(a) $\Delta x = \dfrac{2}{n}$, $f(x) = x^4 - 4x^3 + 4x^2$

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^4 - 4\left(\frac{2i}{n}\right)^3 + 4\left(\frac{2i}{n}\right)^2\right]\frac{2}{n}$$

(b) $$\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^4 - 4\left(\frac{2i}{n}\right)^3 + 4\left(\frac{2i}{n}\right)^2\right]\frac{2}{n} = \left[\frac{8(n+1)(6n^3+9n^2+n-1)}{15n^3} - \frac{8(n+1)^2}{n} + \frac{8(n+1)(2n+1)}{3n}\right]\frac{2}{n} = \left[\frac{8(n+1)(n^3-n^2+n-1)}{15n^3}\right]\frac{2}{n}$$

(c) $$A = \lim_{n\to\infty}\left[\frac{8(n+1)(n^3-n^2+n-1)}{15n^3}\right]\frac{2}{n} = \frac{16}{15}$$

4. $y = \dfrac{1}{2}x^5 + 2x^3$, $[0, 2]$, $c_i = \dfrac{2i}{n}$, $\Delta x = \dfrac{2}{n}$

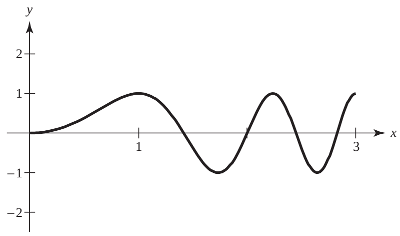
(a) $$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{1}{2}\left(\frac{2i}{n}\right)^5 + 2\left(\frac{2i}{n}\right)^3\right]\frac{2}{n}$$

(b) $$\sum_{i=1}^{n}\left[\frac{1}{2}\left(\frac{2i}{n}\right)^5 + 2\left(\frac{2i}{n}\right)^3\right]\frac{2}{n} = \left[\frac{4(n+1)^2(2n^2+2n-1)}{3n^3} + \frac{4(n+1)^2}{n}\right]\frac{2}{n} = \left[\frac{8(n+1)^2(5n^2+2n-1)}{3n^4}\right]$$

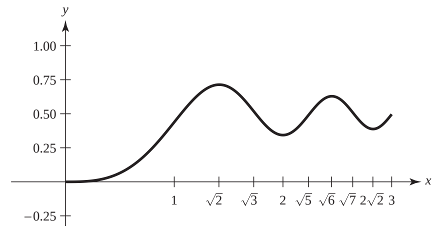
(c) $$A = \lim_{n\to\infty}\left[\frac{8(n+1)^2(5n^2+2n-1)}{3n^4}\right] = \frac{40}{3}$$

5. $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$

(a)



(b)



The zeros of $y = \sin\frac{\pi x^2}{2}$ correspond to the relative extrema of $S(x)$.

(c) $S'(x) = \sin\frac{\pi x^2}{2} = 0 \implies \frac{\pi x^2}{2} = n\pi \implies x^2 = 2n \implies x = \sqrt{2n}$, n integer.

Relative maximum at $x = \sqrt{2} \approx 1.4142$ and $x = \sqrt{6} \approx 2.4495$

Relative minimum at $x = 2$ and $x = \sqrt{8} \approx 2.8284$

(d) $S''(x) = \cos\left(\frac{\pi x^2}{2}\right)(\pi x) = 0 \implies \frac{\pi x^2}{2} = \frac{\pi}{2} + n\pi \implies x^2 = 1 + 2n \implies x = \sqrt{1 + 2n}$, n integer

Points of inflection at $x = 1, \sqrt{3}, \sqrt{5}$, and $\sqrt{7}$.

6. (a) $\int_{-1}^{1} \cos x\, dx \approx \cos\left(-\frac{1}{\sqrt{3}}\right) + \cos\left(\frac{1}{\sqrt{3}}\right) = 2\cos\left(\frac{1}{\sqrt{3}}\right) \approx 1.6758$

$\int_{-1}^{1} \cos x\, dx = \sin x\Big]_{-1}^{1} = 2\sin(1) \approx 1.6829$

Error: $|1.6829 - 1.6758| = 0.0071$

(b) $\int_{-1}^{1} \frac{1}{1 + x^2}\, dx \approx \frac{1}{1 + (1/3)} + \frac{1}{1 + (1/3)} = \frac{3}{2}$

(**Note:** exact answer is $\pi/2 \approx 1.5708$)

(c) Let $p(x) = ax^3 + bx^2 + cx + d$.

$\int_{-1}^{1} p(x)\, dx = \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_{-1}^{1} = \frac{2b}{3} + 2d$

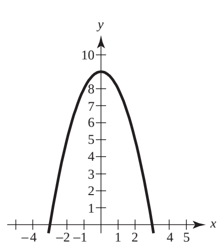
$p\left(-\frac{1}{\sqrt{3}}\right) + p\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{b}{3} + d\right) + \left(\frac{b}{3} + d\right) = \frac{2b}{3} + 2d$

7. (a) Area $= \int_{-3}^{3} (9 - x^2)\, dx = 2\int_0^3 (9 - x^2)\, dx$

$= 2\left[9x - \frac{x^3}{3}\right]_0^3$

$= 2[27 - 9] = 36$

(b) Base $= 6$, height $= 9$, Area $= \frac{2}{3}bh = \frac{2}{3}(6)(9) = 36$



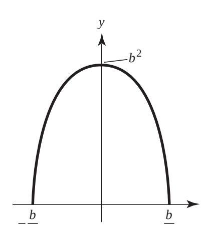
(c) Let the parabola be given by $y = b^2 - a^2x^2$, $a, b > 0$.

Area $= 2\int_0^{b/a} (b^2 - a^2x^2)\, dx$

$= 2\left[b^2x - a^2\frac{x^3}{3}\right]_0^{b/a}$

$= 2\left[b^2\left(\frac{b}{a}\right) - \frac{a^2}{3}\left(\frac{b}{a}\right)^3\right]$

$= 2\left[\frac{b^3}{a} - \frac{1}{3}\frac{b^3}{a}\right] = \frac{4}{3}\frac{b^3}{a}$



Base $= \frac{2b}{a}$, height $= b^2$

Archimedes' Formula: Area $= \frac{2}{3}\left(\frac{2b}{a}\right)(b^2) = \frac{4}{3}\frac{b^3}{a}$

8. Let d be the distance traversed and a be the uniform acceleration. We can assume that $v(0) = 0$ and $s(0) = 0$. Then

$$a(t) = a$$
$$v(t) = at$$
$$s(t) = \frac{1}{2}at^2.$$

$s(t) = d$ when $t = \sqrt{\dfrac{2d}{a}}$.

The highest speed is $v = a\sqrt{\dfrac{2d}{a}} = \sqrt{2ad}$.

The lowest speed is $v = 0$.

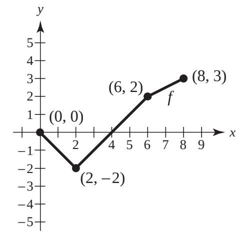
The mean speed is $\dfrac{1}{2}\left(\sqrt{2ad} + 0\right) = \sqrt{\dfrac{ad}{2}}$.

The time necessary to traverse the distance d at the mean speed is

$$t = \frac{d}{\sqrt{ad/2}} = \sqrt{\frac{2d}{a}}$$

which is the same as the time calculated above.

9. (a)



(b)

x	0	1	2	3	4	5	6	7	8
$F(x)$	0	$-\frac{1}{2}$	-2	$-\frac{7}{2}$	-4	$-\frac{7}{2}$	-2	$\frac{1}{4}$	3

(c) $f(x) = \begin{cases} -x, & 0 \le x < 2 \\ x - 4, & 2 \le x < 6 \\ \frac{1}{2}x - 1, & 6 \le x \le 8 \end{cases}$

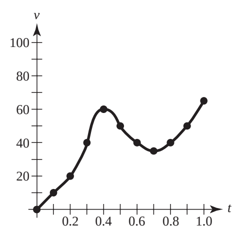
$$F(x) = \int_0^x f(t)\,dt = \begin{cases} (-x^2/2), & 0 \le x < 2 \\ (x^2/2) - 4x + 4, & 2 \le x < 6 \\ (1/4)x^2 - x - 5, & 6 \le x \le 8 \end{cases}$$

$F'(x) = f(x)$. F is decreasing on $(0, 4)$ and increasing on $(4, 8)$. Therefore, the minimum is -4 at $x = 4$, and the maximum is 3 at $x = 8$.

(d) $F''(x) = f'(x) = \begin{cases} -1, & 0 < x < 2 \\ 1, & 2 < x < 6 \\ \frac{1}{2}, & 6 < x < 8 \end{cases}$

$x = 2$ is a point of inflection, whereas $x = 6$ is not.

10. (a)



(b) v is increasing (positive acceleration) on $(0, 0.4)$ and $(0.7, 1.0)$.

(c) Average acceleration $= \dfrac{v(0.4) - v(0)}{0.4 - 0} = \dfrac{60 - 0}{0.4} = 150$ mi/hr^2

(d) This integral is the total distance traveled in miles.

$$\int_0^1 v(t)\,dt \approx \frac{1}{10}[0 + 2(20) + 2(60) + 2(40) + 2(40) + 65] = \frac{385}{10} = 38.5 \text{ miles}$$

(e) One approximation is

$$a(0.8) \approx \frac{v(0.9) - v(0.8)}{0.9 - 0.8} = \frac{50 - 40}{0.1} = 100 \text{ mi/hr}^2$$

(other answers possible)

11. $\int_0^x f(t)(x - t)\,dt = \int_0^x xf(t)\,dt - \int_0^x tf(t)\,dt = x\int_0^x f(t)\,dt - \int_0^x tf(t)\,dt$

Thus, $\frac{d}{dx}\int_0^x f(t)(x - t)\,dt = xf(x) + \int_0^x f(t)\,dt - xf(x) = \int_0^x f(t)\,dt$

Differentiating the other integral,

$$\frac{d}{dx}\int_0^x\left(\int_0^x f(v)\,dv\right)dt = \int_0^x f(v)\,dv.$$

Thus, the two original integrals have equal derivatives,

$$\int_0^x f(t)(x - t)\,dt = \int_0^x\left(\int_0^t f(v)\,dv\right)dt + C.$$

Letting $x = 0$, we see that $C = 0$.

12. Consider $F(x) = [f(x)]^2 \implies F'(x) = 2f(x)f'(x)$. Thus,

$$\begin{aligned}\int_a^b f(x)f'(x)\,dx &= \int_a^b \frac{1}{2}F'(x)\,dx\\ &= \left[\frac{1}{2}F(x)\right]_a^b\\ &= \frac{1}{2}[F(b) - F(a)]\\ &= \frac{1}{2}[f(b)^2 - f(a)^2].\end{aligned}$$

13. Consider $\int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big]_0^1 = \frac{2}{3}$. The corresponding Riemann Sum using right-hand endpoints is

$$\begin{aligned}S(n) &= \frac{1}{n}\left[\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\right]\\ &= \frac{1}{n^{3/2}}\left[\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}\right].\end{aligned}$$

Thus, $\lim_{n\to\infty}\frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n^{3/2}} = \frac{2}{3}$.

14. Consider $\int_0^1 x^5\,dx = \frac{x^6}{6}\Big]_0^1 = \frac{1}{6}$.

The corresponding Riemann Sum using right endpoints is

$$\begin{aligned}S(n) &= \frac{1}{n}\left[\left(\frac{1}{n}\right)^5 + \left(\frac{2}{n}\right)^5 + \cdots + \left(\frac{n}{n}\right)^5\right]\\ &= \frac{1}{n^6}[1^5 + 2^5 + \cdots + n^5].\end{aligned}$$

Thus, $\lim_{n\to\infty} S(n) = \lim_{n\to\infty}\frac{1^5 + 2^5 + \cdots + n^5}{n^6} = \frac{1}{6}$.

15. By Theorem 4.8, $0 < f(x) \le M \implies \int_a^b f(x)\,dx \le \int_a^b M\,dx = M(b - a)$.

Similarly, $m \le f(x) \implies m(b - a) = \int_a^b m\,dx \le \int_a^b f(x)\,dx$.

Thus, $m(b - a) \le \int_a^b f(x)\,dx \le M(b - a)$. On the interval $[0, 1]$, $1 \le \sqrt{1 + x^4} \le \sqrt{2}$ and $b - a = 1$.

Thus, $1 \le \int_0^1 \sqrt{1 + x^4}\,dx \le \sqrt{2}$. $\left(\textbf{Note: } \int_0^1 \sqrt{1 + x^4}\,dx \approx 1.0894\right)$

16. (a) Let $A = \int_0^b \frac{f(x)}{f(x) + f(b-x)}\,dx.$

Let $u = b - x$, $du = -dx$.

$$A = \int_b^0 \frac{f(b-u)}{f(b-u) + f(u)}(-du)$$
$$= \int_0^b \frac{f(b-u)}{f(b-u) + f(u)}\,du$$
$$= \int_0^b \frac{f(b-x)}{f(b-x) + f(x)}\,dx$$

Then,

$$2A = \int_0^b \frac{f(x)}{f(x) + f(b-x)}\,dx + \int_0^b \frac{f(b-x)}{f(b-x) + f(x)}\,dx$$
$$= \int_0^b 1\,dx = b.$$

Thus, $A = \frac{b}{2}.$

(b) $b = 1 \Rightarrow \int_0^1 \frac{\sin x}{\sin(1-x) + \sin x}\,dx = \frac{1}{2}$

(c) $b = 3, f(x) = \sqrt{x}$

$$\int_0^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{3-x}}dx = \frac{3}{2}$$

17. (a) $(1+i)^3 = 1 + 3i + 3i^2 + i^3 \Rightarrow (1+i)^3 - i^3 = 3i^2 + 3i + 1$

(b) $3i^2 + 3i + 1 = (i+1)^3 - i^3$

$$\sum_{i=1}^{n}(3i^2 + 3i + 1) = \sum_{i=1}^{n}[(i+1)^3 - i^3]$$
$$= (2^3 - 1^3) + (3^3 - 2^3) + \cdots + [((n+1)^3 - n^3)]$$
$$= (n+1)^3 - 1$$

Hence, $(n+1)^3 = \sum_{i=1}^{n}(3i^2 + 3i + 1) + 1.$

(c) $(n+1)^3 - 1 = \sum_{i=1}^{n}(3i^2 + 3i + 1) = \sum_{i=1}^{n}3i^2 + \frac{3(n)(n+1)}{2} + n$

$$\Rightarrow \sum_{i=1}^{n}3i^2 = n^3 + 3n^2 + 3n - \frac{3n(n+1)}{2} - n$$
$$= \frac{2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n}{2}$$
$$= \frac{2n^3 + 3n^2 + n}{2}$$
$$= \frac{n(n+1)(2n+1)}{2}$$
$$\Rightarrow \sum_{i=1}^{n}i^2 = \frac{n(n+1)(2n+1)}{6}$$

18. Since $-|f(x)| \le f(x) \le |f(x)|$,

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx \Rightarrow \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

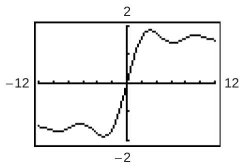
19. (a) $R < I < T < L$

(b) $S(4) = \frac{4-0}{3(4)}[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)]$

$$\approx \frac{1}{3}\left[4 + 4(2) + 2(1) + 4\left(\frac{1}{2}\right) + \frac{1}{4}\right] \approx 5.417$$

20. $\text{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$

(a)



(b) $S_i'(x) = \frac{\sin x}{x}$ $\quad$ $\text{Si}'(x) = 0$ for $x = 2n\pi$

For positive x, $x = (2n - 1)\pi$

For negative x, $x = 2n\pi$

Maxima at $\pi, 3\pi, 5\pi, \ldots$ and $-2\pi, -4\pi, -6\pi, \ldots$

(c) $\text{Si}''(x) = \frac{x\cos x - \sin x}{x^2} = 0$

$x\cos x = \sin x$ for $x \approx 4.4934$

$\text{Si}(4.4934) \approx 1.6556$

(d) Horizontal asymptotes at $y = \pm\frac{\pi}{2}$

$$\lim_{x\to\infty} \text{Si}(x) = \frac{\pi}{2}$$

$$\lim_{x\to-\infty} \text{Si}(x) = \frac{-\pi}{2}$$