51. $y' = \sec^2 x$, so the area is $\int_0^{\pi/4} 2\pi(\tan x)\sqrt{1+\left(\sec^2 x\right)^2}\,dx$, which using NINT evaluates to ≈ 3.84.

52. $x = \frac{1}{y}$ and $x' = -\frac{1}{y^2}$, so the area is $\int_1^2 2\pi\left(\frac{1}{y}\right)\sqrt{1+\left(-\frac{1}{y^2}\right)^2}\,dy$, which using NINT evaluates to $\approx$ 5.02.

53. (a) The two curves intersect at $x = 1.2237831$. Store this value as A.

Area $= \int_0^A (2+\sin x - \sec x)\,dx = 1.366$.

(b) Volume $= \int_0^A \pi\left((2+\sin x)^2 - (\sec x)^2\right)dx = 16.404$.

(c) Volume $= \int_0^A (2+\sin x - \sec x)^2\,dx = 1.629$.

54. (a) Average temp

$= \frac{1}{14-6}\int_6^{14}\left(80 - 10\cos\left(\frac{\pi t}{12}\right)\right)dt \approx 87°F.$

(b) $F(t) = 80 - 10\cos\left(\frac{\pi t}{12}\right) \ge 78$ for

$5.2308694 \le t \le 18.766913$.

Store these two values as A and B.

(c) Cost $= 0.05\int_A^B\left(80 - 10\cos\left(\frac{\pi t}{12}\right) - 78\right)dt \approx 5.10$

The cost was about \$5.10.

55. (a) $\int_9^{17} \frac{15600}{(t^2 - 24t + 160)}\,dt \approx 6004$ people.

(b) $15\int_9^{17} \frac{15600}{(t^2 - 24t + 160)}\,dt + 11\int_{17}^{23} \frac{15600}{(t^2 - 24t + 160)}\,dt \approx 104{,}048$ dollars

(c) $H'(17) = E(17) - L(17) \approx -380$ people. $H(17)$ is the number of people in the park at 5:00, and $H'(17)$ is the rate at which the number of people in the park is changing at 5:00.

(d) When $H'(t) = E(t) - L(t) = 0$; that is, at $t = 15.795$

Chapter 8

Sequences, L'Hôpital's Rule, and Improper Integrals

Section 8.1 Sequences (pp. 435–443)

Quick Review 8.1

1. $f(5) = \frac{5}{5+3} = \frac{5}{8}$

2. $f(-2) = \frac{-2}{-2+3} = -2$

3. $-2 + (3-1)(1.5) = 1$

4. $-7 + (5-1)(3) = 5$

5. $1.5(2^{4-1}) = 12$

6. $-2(1.5^{3-1}) = -4.5$

7. $\lim_{x\to\infty} \frac{5x^3 + 2x^2}{3x^4 + 16x^2} = \lim_{x\to\infty} \frac{5x^3}{3x^4} = 0$

8. $\lim_{x\to 0} \frac{\sin(3x)}{x} = \lim_{x\to 0} \frac{3x}{x} = 3$

9. $\lim_{x\to\infty}\left(x \sin\left(\frac{1}{x}\right)\right) = \lim_{x\to\infty}\left(x\,\frac{1}{x}\right) = 1$

10. $\lim_{x\to\infty} \frac{2x^3 + x^2}{x+1} = \frac{2x^3}{x} =$ Does not exist, or ∞.

Section 8.1 Exercises

1. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}; \frac{50}{51}$

2. $2, \frac{5}{2}, \frac{8}{3}, \frac{11}{4}, \frac{14}{5}, \frac{17}{6}; \frac{149}{50}$

3. $2, \frac{9}{4}, \frac{64}{27}, \frac{625}{256}, \frac{7776}{3125} \approx 2.48832,$

$\frac{117649}{46656} \approx 2.521626, \left(\frac{51}{50}\right)^{50} \approx 2.691588$

4. $-2, -2, 0, 4, 10, 18, 2350$

5. $3, 1, -1, -3; -11$

6. $-2, -1, 0, 1; 5$

7. $2, 4, 8, 16; 256$

8. $10, 11, 12.1, 13.31; 19.487171 \approx 10(1.1)^7$

9. $1, 1, 2, 3; 21$

10. $-3, 2, -1, 1; 2$

11. (a) 3

(b) $a + 7d = -2 + 7(3) = 19$

(c) $a_n = a_{n-1} + 3$

(d) $a_n = -2 + (n-1)(3) = 3n - 5$

12. (a) -2

(b) $a + 7d = 15 + 7(-2) = 1$

(c) $a_n = a_{n-1} - 2$

(d) $a_n = 15 + (n-1)(-2) = -2n + 17$

13. (a) $\frac{1}{2}$

(b) $a + 7d = 1 + 7\left(\frac{1}{2}\right) = \frac{9}{2}$

13. Continued

(c) $a_n = a_{n-1} + \frac{1}{2}$

(d) $a_n = 1 + (n-1)\left(\frac{1}{2}\right) = \frac{(n+1)}{2}$

14. (a) 0.1

(b) $a + 7d = 3 + 7(0.1) = 3.7$

(c) $a_n = a_{n-1} + 0.1$

(d) $a_n = 3 + (n-1)(0.1) = 0.1n + 2.9$

15. (a) $\frac{1}{2}$

(b) $8\left(\frac{1}{2}\right)^8 = 0.03125$

(c) $a_n = \left(\frac{1}{2}\right)a_{n-1}$

(d) $a_n = 8\left(\frac{1}{2}\right)^{n-1} = 2^{4-n}$

16. (a) 1.5

(b) $(1)(1.5)^8 \approx 25.6289$

(c) $a_n = (1.5)a_{n-1}$

(d) $a_n = (1)(1.5)^{n-1} = (1.5)^{n-1}$

17. (a) -3

(b) $(-3)^9 = -19,683$

(c) $a_n = -3a_{n-1}$

(d) $a_n = (-3)(-3)^{n-1} = (-3)^n$

18. (a) -1

(b) $(5)(-1)^8 = 5$

(c) $a_n = -a_{n-1}$

(d) $a_n = 5(-1)^{n-1}$

19. $\frac{7-(-2)}{3} = 3$

$a_1 = -2 - 3 = -5$

$a_n = a_{n-1} + 3$ for all $n \geq 2$

20. $\frac{-3-5}{4} = -2$

$a_1 = 5 - (-2)(4) = 13$

$a_n = 13 + (n-1)(-2) = -2n + 15$

21. $r = \left(\frac{3010000}{3010}\right)^{1/3} = 10$

$a_1 = \frac{3010}{10^3} = 3.01$

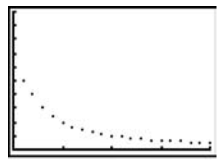
$a_n = 3.01(10)^{n-1}, n \geq 1$

22. $r = \left(\frac{16}{-1/2}\right)^{1/5} = -2$

$a_1 = \frac{-1/2}{-2} = 1/4$

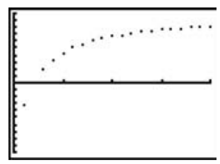
$a_n = (-1)^{n-1}(2)^{n-3}, \ n \geq 1$

23.



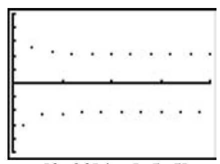
[0, 20] by [0, 1]

24.



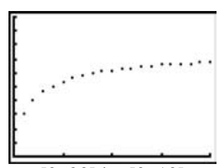
[0, 20] by [–1, 1]

25.



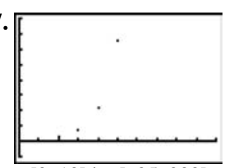
[0, 20] by [–5, 5]

26.



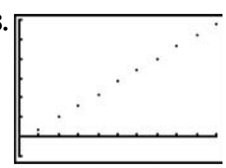
[0, 20] by [0, 10]

27.



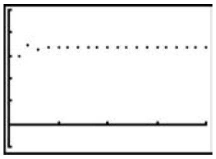
[0, 10] by [–25, 200]

28.



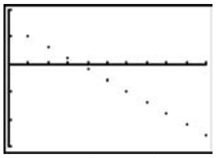
[0, 10] by [–5, 30]

29.



[0, 20] by [–1, 5]

30.



[0, 10] by [–15, 10]

31. $\lim_{n\to\infty} \frac{3n+1}{n} = \lim_{n\to\infty}(3) + \lim_{n\to\infty}\left(\frac{1}{n}\right) = 3 - 0 = 3$

converges, 3

32. $\lim_{n\to\infty} \frac{2n}{n+3} = \lim_{n\to\infty} \frac{2n}{n} = 2$

converges, 2

33. $\lim_{n\to\infty} \frac{2n^2-n-1}{5n^2+n+2} = \lim_{n\to\infty} \frac{2n^2}{5n^2} = \frac{2}{5}$

converges, 2/5

34. $\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0$

converges, 0

35. $n = 2k, \lim_{n\to\infty} (-1)^n \frac{n-1}{n+3} = 1$

$n = 2k-1, \lim_{n\to\infty} (-1)^n \frac{n-1}{n+3} = -1$

diverges

36. $n = 2k, \lim_{n\to\infty} (-1)^n \frac{n+1}{n^2+1} = \lim_{n\to\infty} (-1)^n \frac{n}{n^2} = 0$

$n = 2k-1, \lim_{n\to\infty} (-1)^n \frac{n+1}{n^2+1} = \lim_{n\to\infty} (-1)^n \frac{n}{n^2} = 0$

converges, 0

37. $\lim_{n\to\infty} (1.1)^n = \infty$

diverges

38. $\lim_{n\to\infty} (0.9)^n = 0$

converges, 0

39. $\lim_{n\to\infty} \left(n \sin\left(\frac{1}{n}\right)\right) = \lim_{n\to\infty} \left(n \frac{1}{n}\right) = 1$

converges, 1

40. $\lim_{n\to\infty} \left(\cos\left(\frac{n\pi}{2}\right)\right)$

$-1 \le \cos\left(\frac{n\pi}{2}\right) \le 1$, diverges

41. $\lim_{n\to\infty} \left(\frac{\sin n}{n}\right)$

$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$

$\lim_{n\to\infty} \left(-\frac{1}{n}\right) = \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$

42. $\lim_{n\to\infty} \left(\frac{1}{2^n}\right)$

$\lim_{n\to\infty} \left(-\frac{1}{n}\right) = \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$

$\left(\text{Note: } \frac{1}{2^n} < \frac{1}{n} \text{ for } n \ge 1\right).$

43. $\lim_{n\to\infty} \left(\frac{1}{n!}\right)$

$\lim_{n\to\infty} \left(-\frac{1}{n}\right) = \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$

$\left(\text{Note: } \frac{1}{n!} \le \frac{1}{n} \text{ for } n \ge 1\right)$

44. $\lim_{n\to\infty} \left(\frac{\sin^2 n}{2^n}\right)$

$\lim_{n\to\infty} \left(-\frac{1}{n}\right) = \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$

$\left(\text{Note: } \frac{1}{2^n} < \frac{1}{n} \text{ for } n \ge 1\right)$

45. Graph (b)

46. Graph (c)

47. Table (d)

48. Table (a)

49. False. Consider the sequence with nth term $a_n = -5 + 2(n-1)$. Here $a = -5,\ a_2 = -3,\ a_3 = -1, \text{ and } a_4 = 1$.

50. True. $a_1 > 0,\ r = \frac{a_2}{a_1} > 0$, and

$a_n = a_1 r^{n-1} > 0 \text{ for all } n \ge 2.$

51. C. $\frac{5-(-1)}{2} = 3$

$-1 + 3(5) = 14$

52. E. $\frac{1.25}{2.5} = \frac{1}{2}$

$\frac{2.5}{1/2} = 5$

53. D. $\lim_{n\to\infty} \left(n \sin\left(\frac{3\pi}{n}\right)\right)$

$= \lim_{n\to\infty} \left(n \frac{3\pi}{n}\right) = 3\pi$

54. E. $n = 2k, \lim_{n\to\infty}\left((-1)^n \frac{3n-1}{n+2}\right) = 1$

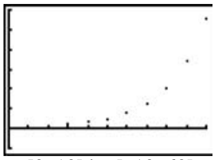
$n = 2k-1, \lim_{n\to\infty}\left((-1)^n \frac{3n-1}{n+2}\right) = -1$

55. (b) $\lim_{n\to\infty}\left(2n\sin\left(\frac{\pi}{n}\right)\right)$

$= \lim_{n\to\infty}\left(2n\left(\frac{\pi}{n}\right)\right) = 2\pi$

56. (a) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55

(b)



[0, 10] by [–10, 60]

57. $a_n = ar^{n-1}$ implies that $\log a_n = \log a + (n-1)\log r$. Thus $\{\log a_n\}$ is an arithmetic sequence with first term log a and common ratio log r.

58. $a_n = a + (n-1)d$ implies that $10^{a_n} = 10^{a+(n-1)d} = 10^a(10^d)^{n-1}$. Thus $\{10^{a_n}\}$ is a geometric sequence with first term 10^a and common ratio 10^d.

59. Given $\varepsilon > 0$ choose $M = \frac{1}{\varepsilon}$. Then $\left|\frac{1}{n} - 0\right| < \varepsilon$ if $n > M$.

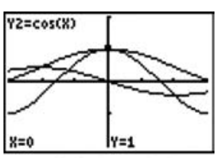
Section 8.2 L'Hôpital's Rule (pp. 444–452)

Exploration 1 Exploring L'Hôpital's Rule Graphically

1. $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = 1$

2. The two graphs suggest that $\lim_{x\to 0}\frac{y_1}{y_2} = \lim_{x\to 0}\frac{y_1'}{y_2'}$.

3. $y_5 = \frac{x\cos x - \sin x}{x^2}$. The graphs of y_3 and y_5 clearly show that l'Hôpital's Rule does not say that $\lim_{x\to 0}\frac{y_1}{y_2}$ is equal to $\lim_{x\to 0}\left(\frac{y_1}{y_2}\right)'$.



[–3, 3] by [–2, 2]

Quick Review 8.2

1.

x	$\left(1+\frac{0.1}{x}\right)^x$
1	1.1000
10	1.1046
100	1.1051
1000	1.1052
10,000	1.1052
1,000,000	1.1052

As $x \to \infty$, $\left(1+\frac{0.1}{x}\right)^x$ approach 1.1052.

2.

x	$x^{1/(\ln x)}$
0.1	2.7183
0.01	2.7183
0.001	2.7183
0.0001	2.7183
0.00001	2.7183

As $x \to 0^+$, $x^{1/(\ln x)}$ approaches 2.7183.

3.

x	$\left(1-\frac{1}{x}\right)^x$
– 1	0.5
– 0.1	0.78679
– 0.01	0.95490
– 0.001	0.99312
– 0.0001	0.99908
– 0.00001	0.99988
– 0.000001	0.99999

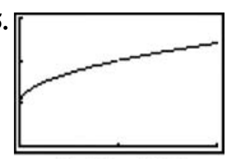
As $x \to 0^-$, $\left(1-\frac{1}{x}\right)^x$ approaches 1.

4.

x	$\left(1+\frac{1}{x}\right)^x$
–1.1	13.981
–1.01	105.77
–1.001	1007.9
–1.0001	10010

As $x \to -1^-$, $\left(1+\frac{1}{x}\right)^x$ goes to ∞.

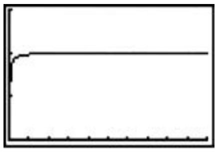
5.



[0, 2] by [0, 3]

As $t \to 1$, $\frac{t-1}{\sqrt{t}-1}$ approaches 2.

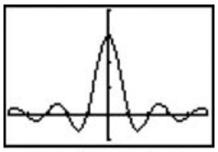
6.



[0, 500] by [0, 3]

As $x \to \infty$, $\dfrac{\sqrt{4x^2+1}}{x+1}$ approaches 2.

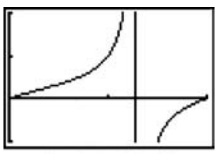
7.



[–5, 5] by [–1, 4]

As $x \to 0$, $\dfrac{\sin 3x}{x}$ approaches 3.

8.



[0, π] by [–1, 2]

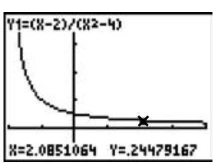
As $\theta \to \dfrac{\pi}{2}$, $\dfrac{\tan\theta}{2+\tan\theta}$ approaches 1.

9. $y = \dfrac{1}{h}\sin h$

10. $y = (1+h)^{1/h}$

Section 8.2 Exercises

1. $\lim_{x\to 2}\left(\dfrac{x-2}{x^2-4}\right)$ appears to be about $\dfrac{1}{4}$;

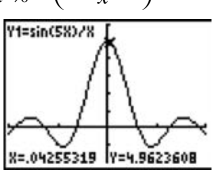


[–2, 4] by [–1, 4]

By L'Hôpital's Rule:

$$\lim_{x\to 2} = \left(\frac{x-2}{x^2-4}\right) = \lim_{x\to 2}\frac{1}{2(2)} = \frac{1}{4}$$

2. $\lim_{x\to 0} = \left(\dfrac{\sin(5x)}{x}\right)$ appears to be about 5;

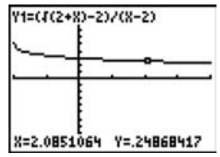


[–2, 2] by [–2, 6]

By L'Hôpital's Rule:

$$\lim_{x\to 0}\left(\frac{\sin(5x)}{x}\right) = \lim_{x\to 0}\left(\frac{5\cos(5(0))}{1}\right) = 5$$

3. $\lim_{x\to 2}\left(\dfrac{\sqrt{2+x}-2}{x-2}\right)$ appears to be about $\dfrac{1}{4}$;

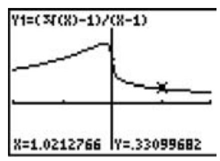


[–2, 4] by [–1, 1]

By L'Hôpital's Rule:

$$\lim_{x\to 2}\left(\frac{\sqrt{2+x}-2}{x-2}\right) = \lim_{x\to 2}\left(\frac{\frac{1}{2}(2+2)^{-1/2}}{1}\right) = \frac{1}{4}$$

4. $\lim_{x\to 1}\left(\dfrac{\sqrt[3]{x}-1}{x-1}\right)$ appears to be about $\dfrac{1}{3}$;



[–2, 2] by [–1, 2]

By L'Hôpital's Rule:

$$\lim_{x\to 1}\left(\frac{\sqrt[3]{x}-1}{x-2}\right) = \lim_{x\to 1}\left(\frac{\frac{1}{3}(1)^{-2/3}}{1}\right) = \frac{1}{3}$$

5. $\lim_{x\to 0}\left(\dfrac{1-\cos x}{x^2}\right) = \lim_{x\to 0}\left(\dfrac{\sin x}{2x}\right) = \lim_{x\to 0}\left(\dfrac{\cos(0)}{2}\right) = \dfrac{1}{2}$

6. $\lim_{\theta\to\pi/2}\left(\dfrac{1-\sin\theta}{1+\cos(2\theta)}\right) = \lim_{\theta\to\pi/2}\left(\dfrac{-\cos\theta}{-2\sin(2\theta)}\right)$

$$= \lim_{\theta\to\pi/2}\left(\frac{-\sin\frac{\pi}{2}}{-4\cos\left(2\frac{\pi}{2}\right)}\right) = \frac{1}{4}$$

7. $\lim_{t\to 0}\left(\dfrac{\cos t-1}{e^t-t-1}\right) = \lim_{t\to 0}\left(\dfrac{-\sin t}{e^t-1}\right) = \lim_{t\to 0}\left(\dfrac{-\cos 0}{e^0}\right) = -1$

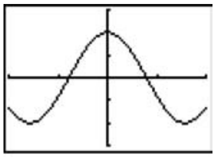
8. $\lim_{x\to 2}\left(\dfrac{x^2-4x+4}{x^3-12x+16}\right) = \lim_{x\to 2}\left(\dfrac{2x-4}{3x^2-12}\right) = \lim_{x\to 2}\left(\dfrac{2}{6(2)}\right) = \dfrac{1}{6}$

9. (a) $\lim_{x\to 0^-}\left(\dfrac{\sin 4x}{\sin 2x}\right) = \lim_{x\to 0^-}\left(\dfrac{4\cos(4(0))}{2\cos(2(0))}\right) = 2$

(b) $\lim_{x\to 0^+}\left(\dfrac{\sin 4x}{\sin 2x}\right) = \lim_{x\to 0^+}\left(\dfrac{4\cos(4(0))}{2\cos(2(0))}\right) = 2$

9. Continued

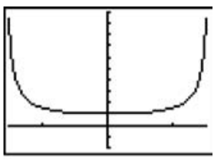
(b)



[–2, 2] by [–3, 3]

10. (a) $\lim_{x\to 0^-}\left(\frac{\tan x}{x}\right)=\lim_{x\to 0^-}\left(\frac{\sec^2 x}{1}\right)=1$

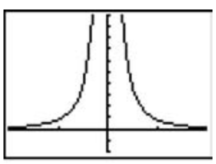
(b) $\lim_{x\to 0^+}\left(\frac{\tan x}{x}\right)=\lim_{x\to 0^+}\left(\frac{\sec^2 x}{1}\right)=1$



[–1.5, 1.5] by [–2, 10]

11. (a) $\lim_{x\to 0^-}\left(\frac{\sin x}{x^3}\right)=\lim_{x\to 0^-}\left(\frac{\cos(0)}{3(0)^2}\right)=\infty$

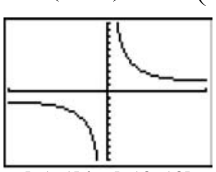
(b) $\lim_{x\to 0^+}\left(\frac{\sin x}{x^3}\right)=\lim_{x\to 0^+}\left(\frac{\cos(0)}{3(0)^2}\right)=\infty$



[–2, 2] by [–2, 10]

12. (a) $\lim_{x\to 0^-}\left(\frac{\tan x}{x^2}\right)=\lim_{x\to 0^-}\left(\frac{\sec^2(0)}{2(-0)}\right)=-\infty$

(b) $\lim_{x\to 0^+}\left(\frac{\tan x}{x^2}\right)=\lim_{x\to 0^+}\left(\frac{\sec^2(0)}{2(+0)}\right)=\infty$



[–1, 1] by [–10, 10]

13. Left:

$\lim_{x\to \pi^-}\left(\frac{\csc x}{1+\cot x}\right)=\frac{\infty}{-\infty}$

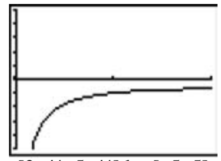
$\lim_{x\to \pi^-}\left(\frac{-\csc x\cot x}{-\csc^2 x}\right)=-1$

Right:

$\lim_{x\to \pi^+}\left(\frac{\csc x}{1+\cot x}\right)=\frac{-\infty}{\infty}$

$\lim_{x\to \pi^+}\left(\frac{-\csc x\cot x}{-\csc^2 x}\right)=-1$

limit = –1



[3π/4, 5π/4] by [–5, 5]

14. Left:

$\lim_{x\to \pi/2^-}\left(\frac{1+\sec x}{\tan x}\right)=\frac{\infty}{\infty}$

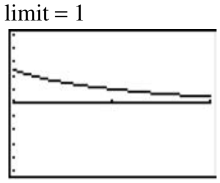
$\lim_{x\to \pi/2^-}\left(\frac{\sec x\tan x}{\sec^2 x}\right)=1$

Right:

$\lim_{x\to \pi/2^+}\left(\frac{1+\sec x}{\tan x}\right)=\frac{-\infty}{-\infty}$

$\lim_{x\to \pi/2^+}\left(\frac{\sec x\tan x}{\sec^2 x}\right)=1$

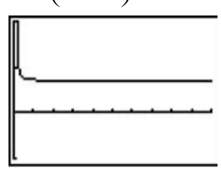
limit = 1



[π/4, 3π/4] by [–5, 5]

15. $\lim_{x\to\infty}\frac{\ln(x+1)}{\log_2 x}=\frac{\infty}{\infty}$

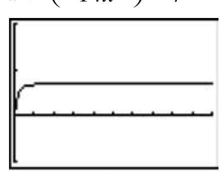
$\lim_{x\to\infty}\left(\frac{\frac{1}{x+1}}{\frac{1}{x\ln 2}}\right)=\ln 2$



[0, 100] by [–1, 2]

16. $\lim_{x\to\infty}\left(\frac{5x^2-3x}{7x^2+1}\right)=\frac{\infty}{\infty}$

$\lim_{x\to\infty}\left(\frac{10x-3}{14x}\right)=\frac{5}{7}$



[0, 100] by [–1, 2]

17. $\lim_{x\to 0^+}(x\ln x)=0\bullet\infty$

$\lim_{x\to 0^+}\left(\frac{\ln x}{1/x}\right)=\lim_{x\to 0^+}\left(\frac{1/x}{-1/x^2}\right)=\lim_{x\to 0^+}(-x)=0$

18. $\lim_{x\to\infty}\left(x\tan\left(\frac{1}{x}\right)\right)=\infty\bullet 0$

$\lim_{x\to\infty}\left(\frac{1}{h}\tan(h)\right)=\lim_{h\to\infty}\left(\frac{1}{h}h\right)=1$

19. $\lim_{x\to 0^+}(\csc x-\cot x+\cos x)=\infty-\infty$

$\lim_{x\to 0^+}\left(\frac{1-\cos x+\cos x\sin x}{\sin x}\right)=\lim_{x\to 0^+}\left(\frac{\sin x-\sin^2 x+\cos^2 x}{\cos x}\right)=1$

20. $\lim_{x\to\infty}(\ln(2x)-\ln(x+1))=\infty-\infty$

~~$\lim_{x\to\infty}\left(\frac{\ln(2x)}{\ln(x+1)}\right)$~~ $=\ln(2)$ $\ln\left(\frac{2x}{x+1}\right)$

21. $\lim_{x\to 0}(e^x+x)^{1/x}=(1+0)^\infty=1^\infty$

$\lim_{x\to 0}\left(\frac{\ln(e^x+x)}{x}\right)=\lim_{x\to 0}\left(\frac{x+\ln x}{x}\right)=\lim_{x\to 0}\left(\frac{1+\frac{1}{x}}{1}\right)=2$

$\lim_{x\to 0}e^{\ln f(x)}=e^2$

22. $\lim_{x\to 1}(x^{1/(x-1)})=1^\infty$

$\lim_{x\to 1}\frac{\ln x}{x-1}=\lim_{x\to 1}\frac{1/x}{1}=1$

$\lim_{x\to 1}e^{\ln f(x)}=e^1=e$

23. $\lim_{x\to 1}(x^2-2x+1)^{x-1}=(1^2-2(1)+1)^{1-1}=0^0$

$\lim_{x\to 1}\frac{\ln(x^2-2x+1)}{1/x-1}=\lim_{x\to 1}\frac{\frac{2}{x-1}}{-1/(x-1)^2}=\lim_{x\to 1}\left(\frac{-2(x-1)^2}{x-1}\right)=0$

$\lim_{x\to 1}e^{\ln f(x)}=e^0=1$

24. $\lim_{x\to 0^+}(\sin x)^x=0^0$

$\lim_{x\to 0^+}\frac{\ln(\sin x)}{1/x}=\lim_{x\to 0^+}\left(\frac{\frac{1}{\tan x}}{-1/x^2}\right)=\lim_{x\to 0^+}\left(-\frac{x^2}{\tan x}\right)$

$=\lim_{x\to 0^+}\left(-\frac{2x}{\sec^2 x}\right)=-\frac{0}{1}=0$

$\lim_{x\to 0^+}e^{\ln f(x)}=e^0=1$

25. $\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x=(1+\infty)^0=\infty^0$

$\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x=\lim_{x\to 0^+}\frac{\ln(1+\frac{1}{x})}{1/x}=\lim_{x\to 0^+}\left(\frac{-\frac{1}{x(x+1)}}{-1/x^2}\right)$

$=\lim_{x\to 0^+}\frac{x^2}{x(x+1)}$

$=\lim_{x\to 0^+}\frac{x}{x+1}=0$

$\lim_{x\to 0^+}e^{\ln f(x)}=e^0=1$

26. $\lim_{x\to\infty}(\ln x)^{1/x}=\infty^\infty$

$\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{1/x}{1}=0$

$\lim_{x\to\infty}e^{\ln f(x)}=e^0=1$

27. (a)

x	10	10^2	10^3	10^4	10^5
$f(x)$	1.1513	0.2303	0.0345	0.00461	0.00058

Estimate the limit to be 0.

(b) $\lim_{x\to\infty}\frac{\ln x^5}{x}=\lim_{x\to\infty}\frac{5\ln x}{x}=\lim_{x\to\infty}\frac{5/x}{1}=\frac{0}{1}=0$

28. (a)

x	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$f(x)$	0.1585	0.1666	0.1667	0.1667	0.1667

Estimate the limit to be $\frac{1}{6}$.

(b) $\lim_{x\to 0^+}\frac{x-\sin x}{x^3}=\lim_{x\to 0^+}\frac{1-\cos x}{3x^2}$

$=\lim_{x\to 0^+}\frac{\sin x}{6x}$

$=\lim_{x\to 0^+}\frac{\cos x}{6}$

$=\frac{1}{6}$

29. Let $f(\theta)=\frac{\sin 3\theta}{\sin 4\theta}$.

θ	$\pm10^0$	$\pm10^{-1}$	$\pm10^{-2}$	$\pm10^{-3}$	$\pm10^{-4}$
$f(\theta)$	–0.1865	0.7589	0.7501	0.7500	0.7500

Estimate the limit to be $=\frac{3}{4}$.

$\lim_{\theta\to 0}\frac{\sin 3\theta}{\sin 4\theta}=\lim_{\theta\to 0}\frac{3\cos 3\theta}{4\cos 4\theta}=\frac{3}{4}$

30. Let $f(t)=\frac{1}{\sin t}-\frac{1}{t}=\frac{t-\sin t}{t\sin t}$.

t	$\pm10^0$	$\pm10^{-1}$	$\pm10^{-2}$	$\pm10^{-3}$
$f(t)$	$\pm$ 0.1884	$\pm$ 0.0167	$\pm$ 0.0017	$\pm$ 0.00017

Estimate the limit to be 0.

$\lim_{t\to 0}\left(\frac{1}{\sin t}-\frac{1}{t}\right)=\lim_{t\to 0}\frac{t-\sin t}{t\sin t}$

$=\lim_{t\to 0}\frac{1-\cos t}{t\cos t+\sin t}$

$=\lim_{t\to 0}\frac{\sin t}{-t\sin t+\cos t+\cos t}=0$

31. Let $f(x) = (1+x)^{1/x}$.

x	10	10^2	10^3	10^4	10^5
$f(x)$	1.2710	1.0472	1.0069	1.0009	1.0001

Estimate the limit to be 1.

$$\ln f(x) = \frac{\ln(1+x)}{x}$$

$$\lim_{x\to\infty} \frac{\ln(1+x)}{x} = \lim_{x\to\infty} \frac{\frac{1}{1+x}}{1} = \frac{0}{1} = 0$$

$$\lim_{x\to\infty} (1+x)^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1$$

32. Let $f(x) = \dfrac{x-2x^2}{3x^2+5x}$.

x	10	10^2	10^3	10^4	10^5
$f(x)$	−0.5429	−0.6525	−0.6652	−0.6665	−0.6667

Estimate the limit to be $-\dfrac{2}{3}$.

$$\lim_{x\to\infty} \frac{x-2x^2}{3x^2+5x} = \lim_{x\to\infty} \frac{1-4x}{6x+5} = \lim_{x\to\infty} -\frac{4}{6} = -\frac{2}{3}.$$

33. $$\lim_{\theta\to 0} \frac{\sin\theta^2}{\theta} = \lim_{\theta\to 0} \frac{2\theta\cos\theta^2}{1} = (2)(0)\cos(0)^2 = 0$$

34. $$\lim_{t\to 1} \frac{t-1}{\ln t - \sin \pi t} = \lim_{t\to 1} \frac{1}{\frac{1}{t} - \pi\cos \pi t} = \frac{1}{1-\pi(-1)} = \frac{1}{\pi+1}$$

35. $$\lim_{x\to\infty} \frac{\log_2 x}{\log_3(x+3)} = \lim_{x\to\infty} \frac{\frac{1}{x\ln 2}}{\frac{1}{(x+3)\ln 3}} = \lim_{x\to\infty} \frac{(x+3)\ln 3}{x\ln 2} = \lim_{x\to\infty} \frac{x\ln 3 + 3\ln 3}{x \ln 2} = \lim_{x\to\infty} \frac{\ln 3}{\ln 2} = \frac{\ln 3}{\ln 2}$$

36. $$\lim_{y\to 0^+} \frac{\ln(y^2+2y)}{\ln y} = \lim_{y\to 0^+} \frac{\frac{2y+2}{y^2+2y}}{\frac{1}{y}} = \lim_{y\to 0^+} \frac{y(2y+2)}{y^2+2y} = \lim_{y\to 0^+} \frac{(2y^2+2y)}{y^2+2y} = \lim_{y\to 0^+} \frac{4y+2}{2y+2} = \frac{4(0)+2}{2(0)+2} = \frac{2}{2} = 1$$

37. $$\lim_{y\to\pi/2} \left(\frac{\pi}{2} - y\right)\tan y = \lim_{y\to\pi/2} \frac{\left(\frac{\pi}{2} - y\right)\sin y}{\cos y} = \lim_{y\to\pi/2} \frac{\left(\frac{\pi}{2} - y\right)\cos y + (-1)\sin y}{-\sin y} = \frac{\left(\frac{\pi}{2} - \frac{\pi}{2}\right)\cos\frac{\pi}{2} + (-1)\sin\frac{\pi}{2}}{-\sin\frac{\pi}{2}} = \frac{(-1)(1)}{-(1)} = 1$$

38. $$\lim_{x\to 0^+} (\ln x - \ln \sin x) = \lim_{x\to 0^+} \ln\frac{x}{\sin x}$$

Let $f(x) = \dfrac{x}{\sin x}$.

$$\lim_{x\to 0^+} \frac{x}{\sin x} = \lim_{x\to 0^+} \frac{1}{\cos x} = 1$$ Therefore,

$$\lim_{x\to 0^+} (\ln x - \ln \sin x) = \lim_{x\to 0^+} \ln f(x) = \ln 1 = 0$$

39. $$\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sqrt{x}}\right) = \lim_{x\to 0^+} \frac{1-\sqrt{x}}{x} = \infty$$

40. The limit leads to the indeterminate form ∞^0.

Let $f(x) = \left(\dfrac{1}{x^2}\right)^x$.

$$\ln\left(\frac{1}{x^2}\right)^x = x\ln\left(\frac{1}{x^2}\right) = \frac{\ln\left(\frac{1}{x^2}\right)}{\frac{1}{x}}$$

$$\lim_{x\to 0} \frac{\ln\left(\frac{1}{x^2}\right)}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{-2/x^3}{1/x^2}}{-1/x^2} = \lim_{x\to 0} 2x = 0$$

$$\lim_{x\to 0} \left(\frac{1}{x^2}\right)^x = \lim_{x\to 0} e^{\ln f(x)} = e^0 = 1$$

41. $$\lim_{x\to\pm\infty} \frac{3x-5}{2x^2-x+2} = \lim_{x\to\pm\infty} \frac{3}{4x-1} = 0$$

42. $\lim_{x\to 0}\dfrac{\sin 7x}{\tan 11x}=\lim_{x\to 0}\dfrac{7\cos 7x}{11\sec^2 11x}=\dfrac{7}{11}$

43. The limit leads to the indeterminate form ∞^0.

Let $f(x)=(1+2x)^{1/(2\ln x)}$.

$$\ln(1+2x)^{1/(2\ln x)}=\frac{\ln(1+2x)}{2\ln x}$$

$$\lim_{x\to\infty}\frac{\ln(1+2x)}{2\ln x}=\lim_{x\to\infty}\frac{\frac{2}{1+2x}}{\frac{2}{x}}=\lim_{x\to\infty}\frac{x}{1+2x}=\lim_{x\to\infty}\frac{1}{2}=\frac{1}{2}$$

$$\lim_{x\to\infty}(1+2x)^{1/(2\ln x)}=\lim_{x\to\infty}e^{\ln f(x)}=e^{1/2}=\sqrt{e}$$

44. The limit leads to the indeterminate form 0^0.

Let $f(x)=(\cos x)^{\cos x}$.

$$\ln(\cos x)^{\cos x}=(\cos x)\ln(\cos x)=\frac{\ln(\cos x)}{\sec x}$$

$$\lim_{x\to\pi/2^-}\frac{\ln(\cos x)}{\sec x}=\lim_{x\to\pi/2^-}\frac{\frac{-\sin x}{\cos x}}{\sec x\tan x}$$

$$=\lim_{x\to\pi/2^-}\frac{-\tan x}{\sec x\tan x}$$

$$=\lim_{x\to\pi/2^-}-\cos x=0$$

$$\lim_{x\to\frac{\pi}{2}^-}=(\cos x)^{\cos x}=\lim_{x=\frac{\pi}{2}^-}e^{\ln f(x)}=e^0=1$$

45. The limit leads to the indeterminate form 1^∞.

Let $f(x)=(1+x)^{1/x}$.

$$\ln(1+x)^{1/x}=\frac{\ln(1+x)}{x}$$

$$\lim_{x\to 0^+}\frac{\ln(1+x)}{x}=\lim_{x\to 0^+}\frac{\frac{1}{1+x}}{1}=1$$

$$\lim_{x\to 0^+}(1+x)^{1/x}=\lim_{x\to 0^+}e^{\ln f(x)}=e^1=e$$

46. The limit leads to the indeterminate form

Let $f(x)=(\sin x)^{\tan x}$

$$\ln(\sin x)^{\tan x}=\tan x\ln(\sin x)=\frac{\ln(\sin x)}{\cot x}$$

$$\lim_{x\to 0^+}\frac{\ln(\sin x)}{\cot x}=\lim_{x\to 0^+}\frac{\frac{\cos x}{\sin x}}{-\csc^2 x}=\lim_{x\to 0^+}(-\sin x\cos x)=0$$

$$\lim_{x\to 0^+}(\sin x)^{\tan x}=\lim_{x\to 0^+}e^{\ln f(x)}=e^0=1$$

47. The limit leads to the indeterminate form $1^{-\infty}$.

Let $f(x)=x^{1/(1-x)}$.

$$\ln x^{1/(1-x)}=\frac{\ln x}{1-x}$$

$$\lim_{x\to 1^+}\frac{\ln x}{1-x}=\lim_{x\to 1^+}\frac{\frac{1}{x}}{-1}=-1$$

$$\lim_{x\to 1^+}x^{1/(1-x)}=\lim_{x\to 1^+}e^{\ln f(x)}=e^{-1}=\frac{1}{e}$$

48. $\int_x^{2x}\dfrac{dt}{t}=\Big[\ln|t|\Big]_x^{2x}=\ln|2x|-\ln|x|=\ln\left|\dfrac{2x}{x}\right|$

$$\lim_{x\to\infty}\int_x^{2x}\frac{dt}{t}=\lim_{x\to\infty}\ln\left|\frac{2x}{x}\right|=\lim_{x\to\infty}\ln 2=\ln 2$$

49. $\lim_{x\to 1}\dfrac{x^3-1}{4x^3-x-3}=\lim_{x\to 1}\dfrac{3x^2}{12x^2-1}=3/11$

50. $\lim_{x\to\infty}\dfrac{2x^2+3x}{x^3+x+1}=\lim_{x\to\infty}\dfrac{4x+3}{3x^2+1}=\lim_{x\to\infty}\dfrac{4}{6x}=0$

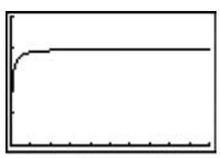
51. $\lim_{x\to 1}\dfrac{\int_1^x\cos t\,dt}{x^2-1}=\lim_{x\to 1}\dfrac{\sin x-\sin 1}{x^2-1}=\lim_{x\to 1}\dfrac{\cos x}{2x}=\dfrac{\cos 1}{2}$

52. $\lim_{x\to 1}\dfrac{\int_1^x\frac{dt}{t}}{x^3-1}=\lim_{x\to 1}\dfrac{\ln x-\ln 1}{x^3-1}=\lim_{x\to 1}\dfrac{1/x}{3x^2}=1/3$

53. (a) L'Hôpital's Rule does not help because applying L'Hôpital's Rule to this quotient essentially "inverts" the problem by interchanging the numerator and denominator (see below). It is still essentially the same problem and one is no closer to a solution. Applying L'Hôpital's Rule a second time returns to the original problem.

$$\lim_{x\to\infty}\frac{\sqrt{9x+1}}{\sqrt{x+1}}=\lim_{x\to\infty}\frac{(9/2)(9x+1)^{-1/2}}{(1/2)(x+1)^{-1/2}}=\lim_{x\to\infty}\frac{9\sqrt{x+1}}{\sqrt{9x+1}}$$

(b)



[0, 100] by [0, 4]

The limit appears to be 3.

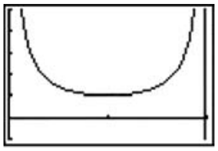
(c) $\lim_{x\to\infty}\dfrac{\sqrt{9x+1}}{\sqrt{x+1}}=\lim_{x\to\infty}\dfrac{\sqrt{9+\frac{1}{x}}}{\sqrt{1+\frac{1}{x}}}=\dfrac{\sqrt{9}}{\sqrt{1}}=3$

54. (a) L'Hôpital's Rule does not help because applying L'Hôpital's Rule to this quotient essentially "inverts" the problem by interchanging the numerator and denominator (see below). It is still essentially the same problem and one is no closer to a solution. Applying L'Hôpital's Rule a second time returns to the original problem.

$$\lim_{x\to\pi/2}\frac{\sec x}{\tan x}=\lim_{x\to\pi/2}\frac{\sec x\tan x}{\sec^2 x}=\lim_{x\to\pi/2}\frac{\tan x}{\sec x}$$

54. Continued

(b)



[0, π] by [−1, 5]

The limit appears to be 1.

(c) $\lim_{x\to\pi/2}\frac{\sec x}{\tan x}=\lim_{x\to\pi/2}\frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}}=\lim_{x\to\pi/2}\frac{1}{\sin x}=1$

55. Find c such that $\lim_{x\to 0} f(x)=c$.

$$\lim_{x\to0}f(x)=\lim_{x\to0}\frac{9x-3\sin 3x}{5x^3}$$
$$=\lim_{x\to0}\frac{9-9\cos 3x}{15x^2}$$
$$=\lim_{x\to0}\frac{27\sin 3x}{30x}$$
$$=\lim_{x\to0}\frac{81\cos 3x}{30}=\frac{81}{30}=\frac{27}{10}$$

Thus, $c=\frac{27}{10}$. This works since $\lim_{x\to0} f(x)=c=f(0)$, so f is continuous.

56. $f(x)$ is defined at $x\neq 0$. $\lim_{x\to0} f(x)$ leads to the indeterminate form 0^0.

$$\ln|x|^x = x\ln|x| = \frac{\ln|x|}{\frac{1}{x}}$$

$$\lim_{x\to0}\frac{\ln|x|}{\frac{1}{x}}=\lim_{x\to0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to0}-x=0$$

$$\lim_{x\to0}|x|^x=\lim_{x\to0}e^{x\ln|x|}=e^0=1$$

Thus, f has a removable discontinuity at $x=0$. Extend the definition of f by letting $f(0)=1$.

57. (a) The limit leads to the indeterminate form 1^∞.

Let $f(k)=\left(1+\frac{r}{k}\right)^{kt}$.

$$\ln f(k)=kt\ln\left(1\frac{r}{k}\right)=\frac{t\ln\left(1+\frac{r}{k}\right)}{\frac{1}{k}}$$

$$\lim_{k\to\infty}\frac{t\ln\left(1+\frac{r}{k}\right)}{\frac{1}{k}}=\lim_{k\to\infty}\frac{t\left(-\frac{r}{k^2}\right)\left(1+\frac{r}{k}\right)^{-1}}{-\frac{1}{k^2}}$$

$$=\lim_{k\to\infty}\frac{rt}{1+\frac{r}{k}}=\frac{rt}{1}=rt$$

$$\lim_{k\to\infty}A_0\left(1+\frac{r}{k}\right)^{kt}=A_0\lim_{k\to\infty}\left(1+\frac{r}{k}\right)^{kt}$$
$$=A_0\lim_{k\to\infty}e^{\ln f(k)}$$
$$=A_0e^{rt}$$

(b) Part (a) shows that as the number of compoundings per year increases toward infinity, the limit of interest compounded k times per year is interest compounded continuously.

58. (a) For $x\neq0$, $\frac{f'(x)}{g'(x)}=\frac{1}{1}=1$.

$$\lim_{x\to0}\frac{f'(x)}{g'(x)}=1$$

$$\lim_{x\to0}\frac{f(x)}{g(x)}=\frac{2}{1}=2$$

(b) This does not contradict L'Hôpital's Rule since $\lim_{x\to0}f(x)=2$ and $\lim_{x\to0}g(x)=1$.

59. (a) $A(t)=\int_0^t e^{-x}dx=\left[-e^{-x}\right]_0^t=e^{-t}+1$

$$\lim_{t\to\infty}A(t)=\lim_{t\to\infty}(-e^{-1}+1)=\lim_{t\to\infty}\left(-\frac{1}{e^t}+1\right)=1$$

(b) $V(t)=\pi\int_0^t(e^{-x})^2\,dx$

$$=\pi\int_0^t e^{-2x}\,dx$$
$$=\pi\left[-\frac{1}{2}e^{-2x}\right]_0^t$$
$$=\pi\left(-\frac{1}{2}e^{-2t}+\frac{1}{2}\right)$$
$$=\frac{\pi}{2}\left(-e^{-2t}+1\right)$$

$$\lim_{t\to\infty}\frac{V(t)}{A(t)}=\lim_{t\to\infty}\frac{\frac{\pi}{2}(-e^{-2t}+1)}{-e^{-t}+1}=\frac{\frac{\pi}{2}(1)}{1}=\frac{\pi}{2}$$

(c) $\lim_{t\to0^+}\frac{V(t)}{A(t)}=\lim_{t\to0^+}\frac{\frac{\pi}{2}(-e^{-2t}+1)}{-e^{t}+1}$

$$=\lim_{t\to0^+}\frac{\frac{\pi}{2}(2e^{-2t})}{e^{-t}}$$
$$=\frac{\frac{\pi}{2}(2)}{1}=\pi$$

60. (a)

x	$f(x)$
0.1	0.04542
0.01	0.00495
0.00	0.00050
0.0001	0.00005

The limit appears to be 0.

(b) $\lim_{x\to 0}\frac{\sin x}{1+2x}=\frac{0}{1}=0$

L'Hôpital's Rule is not applied here because the limit is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, since the denominator has limit 1.

61. (a) $f(x)=e^{x\ln(1+1/x)}$

$1+\frac{1}{x}>0$ when $x<-1$ or $x>0$

Domain: $(-\infty,-1)\cup(0,\infty)$

(b) The form is 0^{-1}, so $\lim_{x\to -1^-} f(x)=\infty$

(c) $\lim_{x\to-\infty} x\ln\left(1+\frac{1}{x}\right)=\lim_{x\to-\infty}\frac{\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$

$$=\lim_{x\to-\infty}\frac{\left(-\frac{1}{x^2}\right)\left(1+\frac{1}{x}\right)^{-1}}{\frac{-1}{x^2}}$$

$$=\lim_{x\to-\infty}\frac{1}{1+\frac{1}{x}}=1$$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty} e^{x\ln(1+1/x)}=e$$

62. False. Need $g'(a)\neq 0$. Consider $f(x)=\sin^2 x$ and $g(x)=x^2$ with $a=0$. Here $\lim_{x\to 0} f'(x)=\lim_{x\to 0} g'(x)=0$.

63. False. The limit is 1.

64. C. $\lim_{x\to 0}\frac{x}{\tan x}=\frac{1}{\sec^2 x}=\frac{1}{1}=1$

65. D. $\lim_{x\to 1}\frac{1-\frac{1}{x}}{1-\frac{1}{x^2}}=\lim_{x\to 1}\frac{\frac{1}{x^2}}{\frac{2}{x^3}}=\lim_{x\to 1}\frac{x^3}{2x^2}=1/2$

66. B. $\lim_{x\to\infty}\frac{\log_2 x}{\log_3 x}=\lim_{x\to\infty}\frac{\frac{1}{x\ln 2}}{\frac{1}{x\ln 3}}=\frac{\ln 3}{\ln 2}$

67. E. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x}=\lim_{x\to\infty}\left(\frac{\ln\left(1+\frac{1}{x}\right)}{1/3x}\right)=\lim_{x\to\infty}-\frac{\frac{1}{x(x+1)}}{-1/3x^2}$

$$=\lim_{x\to\infty}\frac{3x^2}{x(x+1)}=\lim_{x\to\infty}\frac{3}{1}=3$$

$$\lim_{x\to\infty} e^{\ln f(x)}=e^3$$

68. Possible answers:

(a) $f(x)=7(x-3);\ g(x)=x-3$

$$\lim_{x\to 3}\frac{f(x)}{g(x)}=\lim_{x\to 3}\frac{7(x-3)}{x-3}\lim_{x\to 3}\frac{7}{1}=7$$

(b) $f(x)=(x-3)^2;\ g(x)=x-3$

$$\lim_{x\to 3}\frac{f(x)}{g(x)}=\lim_{x\to 3}\frac{(x-3)^2}{x-3}\lim_{x\to 3}\frac{2(2x-3)}{1}=0$$

(c) $f(x)=x-3;\ g(x)=(x-3)^3$

$$\lim_{x\to 3}\frac{f(x)}{g(x)}=\lim_{x\to 3}\frac{x-3}{(x-3)^3}=\lim_{x\to 3}\frac{1}{3(x-3)^2}=\infty$$

69. Answers may vary.

(a) $f(x)=3x+1;\ g(x)=x$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{3x+1}{x}=\lim_{x\to\infty}\frac{3}{1}=3$$

(b) $f(x)=x+1;\ g(x)=x^2$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x+1}{x^2}=\lim_{x\to\infty}\frac{1}{2x}=0$$

(c) $f(x)=x^2;\ g(x)=x+1$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x^2}{x+1}=\lim_{x\to\infty}\frac{2x}{1}=\infty$$

70. (a) Because the difference in the numerator is so small compared to the values being subtracted, any calculator or computer with limited precision will give the incorrect result that $1-\cos x^6$ is 0 for even moderately small values of x. For example, at $x=0.1$, $\cos x^6\approx 0.9999999999995$ (13 places), so on a 10-place calculator, $\cos x^6=1$ and $1-\cos x^6=0$.

(b) Same reason as in part (a) applies.

(c) $\lim_{x\to 0}\frac{1-\cos x^6}{x^{12}}=\lim_{x\to 0}\frac{6x^5\sin x^6}{12x^{11}}$

$$=\lim_{x\to 0}\frac{\sin x^6}{2x^6}$$

$$=\lim_{x\to 0}\frac{6x^5\cos x^6}{12x^5}$$

$$=\lim_{x\to 0}\frac{\cos x^6}{2}=\frac{1}{2}$$

70. Continued

(d) The graph and/or table on a grapher show the value of the function to be 0 for x-values moderately close to 0, but the limit is 1/2. The calculator is giving unreliable information because there is significant round-off error in computing values of this function on a limited precision device.

71. (a) $f'(x)=3x^2,\ g'(x)=2x-1$

$$f(1)-f(-1)=2,\ g(1)-g(-1)=-2$$
$$\frac{3c^2}{2c-1}=\frac{2}{-2}$$
$$3c^2=-2c+1$$
$$3c^2+2c-1=0$$
$$(3c-1)(c+1)=0$$
$$c=\frac{1}{3}\text{ or }c=-1$$

The value of c that satisfies the property is $c=\frac{1}{3}$.

(b) $f'(x)=-\sin x,\ g'(x)=\cos x$

$$f\left(\frac{\pi}{2}\right)-f(0)=-1,\ g\left(\frac{\pi}{2}\right)-g(0)=1$$
$$\frac{-\sin c}{\cos c}=\frac{-1}{1}$$
$$\tan c=1$$
$$c=\tan^{-1}1=\frac{\pi}{4}\text{ on }\left(0,\frac{\pi}{2}\right)$$

72. (a) $\ln f(x)^{g(x)}=g(x)\ln f(x)$

$$\lim_{x\to c}(g(x)\ln f(x))=\left(\lim_{x\to c}g(x)\right)\left(\lim_{x\to c}\ln f(x)\right)=\infty(-\infty)=-\infty$$
$$\lim_{x\to c}f(x)^{g(x)}=\lim_{x\to c}e^{\ln f(x)^{g(x)}}=e^{-\infty}=0$$

(b) $\lim_{x\to c}\left(g(x)\ln f(x)\right)=\left(\lim_{x\to c}g(x)\right)\left(\lim_{x\to c}\ln f(x)\right)=(-\infty)(-\infty)=\infty$

$$\lim_{x\to c}f(x)^{g(x)}=\lim_{x\to c}e^{\ln f(x)^{g(x)}}=e^{\infty}=\infty$$

Quick Quiz Sections 8.1 and 8.2

1. C. $\lim_{x\to 0}\frac{(x+1)^{4/3}-(4/3)x-1}{x^2}$

$$=\lim_{x\to 0}\frac{4/3(x+1)^{1/3}-(4/3)}{2x}$$
$$=\lim_{x\to 0}\frac{4/9(x+1)^{-2/3}}{2}$$
$$=\frac{2}{9}$$

2. D. $\lim_{x\to 0^+}(3x^{2x})$

$$=\lim_{x\to 0^+}\frac{\ln x}{1/2x}$$
$$=\lim_{x\to 0^+}\frac{1/x}{1/2x^2}=0$$
$$\lim_{x\to 0}3e^{\ln f(x)}=3e^0=3$$

3. B. $\lim_{x\to 2}\frac{\int_2^x \sin t\,dt}{x^2-4}$

$$=\lim_{x\to 2}\frac{\cos x-\cos 2}{x^2-4}$$
$$=\lim_{x\to 2}\frac{\sin x}{2x}=\frac{\sin 2}{4}$$

4. (a) $\left(\frac{1/2}{-4}\right)^{1/3}=-\frac{1}{2}$

$$\frac{-4}{-1/2}=8$$

(b) $-\frac{1}{2}$

(c) $a_n=8\left(-\frac{1}{2}\right)^n=(-1)^{n-1}(2^{4-n})$

(d) $a_n=\left(-\frac{1}{2}\right)a_{n-1}$

Section 8.3 Relative Rates of Growth (pp. 453–458)

Exploration 1 Comparing Rates of Growth as $x\to\infty$

1. $\lim_{x\to\infty}\frac{a^x}{x^2}=\lim_{x\to\infty}\frac{(\ln a)(a^x)}{2x}=\lim_{x\to\infty}\frac{(\ln a)^2a^x}{2}=\infty$, so a^x grows faster than x^2 as $x\to\infty$.

2. $\lim_{x\to\infty}\frac{3^x}{2^x}=\lim_{x\to\infty}1.5^x=\infty$

3. $\lim_{x\to\infty}\frac{a^x}{b^x}=\lim_{x\to\infty}\left(\frac{a}{b}\right)^x=\infty$ because $\frac{a}{b}>1$.

Quick Review 8.3

1. $\lim_{x\to\infty}\frac{\ln x}{e^x}=\lim_{x\to\infty}\frac{\frac{1}{x}}{e^x}=0$

2. $\lim_{x\to\infty}\frac{e^x}{x^3}=\lim_{x\to\infty}\frac{e^x}{3x^2}=\lim_{x\to\infty}\frac{e^x}{6x}=\lim_{x\to\infty}\frac{e^x}{6}=\infty$

3. $\lim_{x\to-\infty} \frac{x^2}{e^{2x}} = \infty$

4. $\lim_{x\to\infty} \frac{x^2}{e^{2x}} = \lim_{x\to\infty} \frac{2x}{2e^{2x}} = \lim_{x\to\infty} \frac{2}{4e^{2x}} = 0$

5. $-3x^4$

6. $\frac{2x^3}{x} = 2x^2$

7. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x+\ln x}{x} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{1} = 1$

8. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\sqrt{4x^2+5x}}{2x} = \lim_{x\to\infty} \sqrt{1+\frac{5}{4x}} = 1$

9. (a) $f(x) = \frac{e^x + x^2}{e^x} = 1 + \frac{x^2}{e^x}$

$f'(x) = \frac{2xe^x - x^2e^x}{e^{2x}} = \frac{2x - x^2}{e^x}$

$\frac{2x - x^2}{e^x} = 0$

$x(2-x) = 0$

$x = 0$ or $x = 2$

$f'(x) < 0$ for $x < 0$ or $x > 2$

The graph decreases, increases, and then decreases.

$f(0) = 1; f(2) = 1 + \frac{4}{e^2} \approx 1.541$

f has a local maximum at ≈ (2, 1.541) and has a local minimum at (0, 1).

(b) f is increasing on [0, 2]

(c) f is decreasing on $(-\infty, 0]$ and $[2, \infty)$.

10. $f(x) = \frac{x+\sin x}{x} = 1 + \frac{\sin x}{x}, x \neq 0$

Observe that $\left|\frac{\sin x}{x}\right| < 1$ since $|\sin x| < |x|$ for $x \neq 0$.

$\lim_{x\to0} f(x) = 1 + \lim_{x\to0} \frac{\sin x}{x} = 1 + 1 = 2$

Thus the values of f get close to 2 as x gets close to 0, so f doesn't have an absolute maximum value. f is not defined at 0.

Section 8.3 Exercises

1. $\lim_{x\to\infty} \frac{e^x}{x^3-3x+1} = \lim_{x\to\infty} \frac{e^x}{3x^2-3} = \lim_{x\to\infty} \frac{e^x}{6^x} = \lim_{x\to\infty} \frac{e^x}{6} = \infty$

2. $\lim_{x\to\infty} \frac{e^x}{x^{20}} = \lim_{x\to\infty} \frac{e^x}{20!} = \infty$

3. $\lim_{x\to\infty} \frac{e^x}{e^{\cos x}} = \lim_{x\to\infty} \frac{e^x}{-\sin x\, e^{\cos x}}, -1 \leq \cos x \leq 1,$

$\lim_{x\to\infty} \frac{e^x}{-\sin x\, e^{\cos y}} = \infty$

4. $\lim_{x\to\infty} \frac{e^x}{(5/2)^x} = \frac{e^x}{x!(5/2)} = \infty$

5. $\lim_{x\to\infty} \frac{\ln x}{x - \ln x} = \lim_{x\to\infty} \frac{1/x}{x - 1/x} = \lim_{x\to\infty} \frac{1}{x^2-1} = 0$

6. $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/2(x)^{-1/2}} = \frac{1}{1/2(x)^{-1/2}x} = 0$

7. $\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{1/x}{1/3(x)^{-2/3}} = \lim_{x\to\infty} \frac{1}{1/3(x)^{-2/3}x} = 0$

8. $\lim_{x\to\infty} \frac{\ln x}{x^3} = \lim_{x\to\infty} \frac{1/x}{3x^2} = \lim_{x\to\infty} \frac{1}{3x^3} = 0$

9. $\lim_{x\to\infty} \frac{x^2+4x}{x^2} = \lim_{x\to\infty} \frac{2x+4}{2x} = \lim_{x\to\infty} \frac{2}{2} = 1$

10. $\lim_{x\to\infty} \frac{\sqrt{x^4+5x}}{x^2} = \lim_{x\to\infty} \frac{x^4+5x}{x^4} = \lim_{x\to\infty} \frac{12x^2}{12x^2} = 1$

11. $\lim_{x\to\infty} \frac{(x^6+x^2)^{1/3}}{x^2} = \lim_{x\to\infty} \frac{x^6+x^2}{x^6} = \lim_{x\to\infty} \frac{120x^3}{120x^3} = 1$

12. $\lim_{x\to\infty} \frac{x^2+\sin x}{x^2} = \lim_{x\to\infty} \frac{2x+\cos x}{2x}, -1 \leq \cos x \leq 1, \lim_{x\to\infty} \frac{2x}{2x} = 1$

13. $\lim_{x\to\infty} \frac{\log\sqrt{x}}{\ln x} = \frac{\frac{1}{\frac{1}{2}x\ln 10}}{1/x} = \frac{1}{2\ln 10}$

14. $\lim_{x\to\infty} \frac{e^{x+1}}{e^x} = e$

15. First observe that $\sqrt{1+x^4}$ grows at the same rate as x^2.

$\lim_{x\to\infty} \frac{\sqrt{1+x^4}}{x^2} = \lim_{x\to\infty} = \frac{\sqrt{1+x^4}}{x^4} = \lim_{x\to\infty} \sqrt{\frac{1}{x^4}+1} = 1$

Next compare x^2 with e^x.

$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$

x^2 grows slower than e^x as $x \to \infty$, so $\sqrt{1+x^4}$ grows slower than e^x as $x \to \infty$.

16. $\lim_{x\to\infty} \frac{4^x}{e^x} = \lim_{x\to\infty} \left(\frac{4}{e}\right)^x = \infty$ since $\frac{4}{e} > 1$.

4^x grows faster than e^x as $x \to \infty$.

17. $\lim_{x\to\infty} \frac{x\ln x - x}{e^x} = \lim_{x\to\infty} \frac{x\left(\frac{1}{x}\right) + \ln x - 1}{e^x}$

$= \lim_{x\to\infty} \frac{\ln x}{e^x}$

$= \lim_{x\to\infty} \frac{1/x}{e^x} = 0$

$x \ln x - x$ grows slower than e^x as $x \to \infty$.

18. $\lim_{x\to\infty} \frac{xe^x}{e^x} = \lim_{x\to\infty} x = \infty$

xe^x grows faster than e^x as $x \to \infty$.

19. $\lim_{x\to\infty} \frac{x^{1000}}{e^x} = 0$ (Repeated application of L'Hôpital's Rule gets $\lim_{x\to\infty} \frac{1000!}{e^x} = 0$.)

x^{1000} grows slower than e^x as $x \to \infty$.

20. $\lim_{x\to\infty} \frac{(e^x + e^{-x})/2}{e^x} = \lim_{x\to\infty}\left(\frac{1}{2} + \frac{1}{2e^{2x}}\right) = \frac{1}{2}$

$\frac{e^x + e^{-x}}{2}$ grows at the same rate as e^x as $x \to \infty$.

21. $\lim_{x\to\infty} \frac{x^3+3}{x^2} = \lim_{x\to\infty}\left(x + \frac{3}{x^2}\right) = \infty$

$x^3 + 3$ grows faster than x^2 as $x \to \infty$.

22. $\lim_{x\to\infty} \frac{15x+3}{x^2} = \lim_{x\to\infty}\left(\frac{15}{x} + \frac{3}{x^2}\right) = 0$

$15x + 3$ grows slower than x^2 as $x \to \infty$.

23. $\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$

$\ln x$ grows slower than x^2 as $x \to \infty$.

24. $\lim_{x\to\infty} \frac{2^x}{x^2} = \lim_{x\to\infty} \frac{(\ln 2)2^x}{2x} = \lim_{x\to\infty} \frac{(\ln 2)^2\, 2^x}{2} = \infty.$

2^x grows faster than x^2 as $x \to \infty$.

25. $\lim_{x\to\infty} \frac{\log_2 x^2}{\ln x} = \lim_{x\to\infty} \frac{2\log_2 x}{\ln x} = \lim_{x\to\infty} \frac{2(\ln x)/(\ln 2)}{\ln x} = \frac{2}{\ln 2}$

$\log_2 x^2$ grows at the same rate as $\ln x$ as $x \to \infty$.

26. $\lim_{x\to\infty} \frac{1/\sqrt{x}}{\ln x} = \lim_{x\to\infty} \frac{1}{\sqrt{x}\ln x} = 0$

$\frac{1}{\sqrt{x}}$ grows slower than $\ln x$ as $x \to \infty$.

27. $\lim_{x\to\infty} \frac{e^{-x}}{\ln x} = \lim_{x\to\infty} \frac{1}{e^x \ln x} = 0$

e^{-x} grows slower than $\ln x$ as $x \to \infty$.

28. $\lim_{x\to\infty} \frac{5\ln x}{\ln x} = 5$

$5\ln x$ grows at the same rate as $\ln x$ as $x \to \infty$,

29. Compare e^x to x^x.

$\lim_{x\to\infty} \frac{e^x}{x^x} = \lim_{x\to\infty}\left(\frac{e}{x}\right)^x = 0$

e^x grows slower than x^x.

Compare e^x to $(\ln x)^x$.

$\lim_{x\to\infty} \frac{e^x}{(\ln x)^x} = \lim_{x\to\infty}\left(\frac{e}{\ln x}\right)^x = 0$

e^x grows slower than $(\ln x)^x$.

Compare $x\, e^x$ to $e^{x/2}$.

$\lim_{x\to\infty} \frac{e^x}{e^{x/2}} = \lim_{x\to\infty} e^{x/2} = \infty$

e^x grows faster than $e^{x/2}$.

Compare x^x to $(\ln x)^x$.

$\lim_{x\to\infty} \frac{x^x}{(\ln x)^x} = \lim_{x\to\infty}\left(\frac{x}{\ln x}\right)^x = \infty$ since $\lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{x}{1/x} = \infty.$

x^x grows faster than $(\ln x)^x$.

Thus, in order from slowest-growing to fastest-growing, we get $e^{x/2}$, e^x, $(\ln x)^x$, x^x.

30. Compare 2^x to x^2

$\lim_{x\to\infty} \frac{2^x}{x^2} = \lim_{x\to\infty} \frac{(\ln 2)2^x}{2x} = \lim_{x\to\infty} \frac{(\ln 2)^2\, 2^x}{2} = \infty$

2^x grows faster than x^2.

Compare 2^x to $(\ln 2)^x$.

$\lim_{x\to\infty} \frac{2^x}{(\ln 2)^x} = \lim_{x\to\infty}\left(\frac{2}{\ln 2}\right)^x = \infty$ since $\frac{2}{\ln 2} > 1.$

2^x grows faster than $(\ln 2)^x$

Compare 2^x to e^x.

$\lim_{x\to\infty} \frac{2^x}{e^x} = \lim_{x\to\infty}\left(\frac{2}{e}\right)^x = 0$ since $\frac{2}{e} < 1.$

2^x grows slower than e^x.

30. Continued

Compare x^2 to $(\ln 2)^x$.

$$\lim_{x\to\infty} \frac{x^2}{(\ln 2)^x} = \infty \text{ since } \lim_{x\to\infty} x^2 = \infty \text{ and } \lim_{x\to\infty} (\ln 2)^x = 0.$$

x^2 grows faster than $(\ln 2)^x$.

Thus, in order from slowest-growing to fastest-growing, we get $(\ln 2)^x,\ x^2,\ 2^x,\ e^x$.

31. Compare f_1 to f_2.

$$\lim_{x\to\infty} \frac{f_2(x)}{f_1(x)} = \lim_{x\to\infty} \frac{\sqrt{10x+1}}{\sqrt{x}} = \lim_{x\to\infty} \sqrt{10+\frac{1}{x}} = \sqrt{10}$$

Thus f_1 and f_2 grow at the same rate.

Compare f_1 to f_3.

$$\lim_{x\to\infty} \frac{f_3(x)}{f_1(x)} = \lim_{x\to\infty} \frac{\sqrt{x+1}}{\sqrt{x}} = \lim_{x\to\infty} \sqrt{1+\frac{1}{x}} = 1$$

Thus f_1 and f_3 grow at the same rate.

By transitivity, f_2 and f_3 grow at the same rate, so all three functions grow at the same rate as $x \to \infty$.

32. Compare f_1 to f_2.

$$\lim_{x\to\infty} \frac{f_2(x)}{f_1(x)} = \lim_{x\to\infty} \frac{\sqrt{x^4+x}}{x^2} = \lim_{x\to\infty} \sqrt{1+\frac{1}{x^3}} = 1$$

Thus f_1 to f_3 grow at the same rate.

Compare f_1 to f_2.

$$\lim_{x\to\infty} \frac{f_3(x)}{f_1(x)} = \lim_{x\to\infty} \frac{\sqrt{x^4-x^3}}{x^2} = \lim_{x\to\infty} \sqrt{1-\frac{1}{x}} = 1$$

Thus f_1 and f_3 grow at the same rate.

By transitivity, f_2 and f_3 grow at the same rate, so all three functions grow at the same rate as $x \to \infty$.

33. Compare f_1 to f_2.

$$\lim_{x\to\infty} \frac{f_2(x)}{f_1(x)} = \lim_{x\to\infty} \frac{\sqrt{9^x+2^x}}{3^x}$$

$$= \lim_{x\to\infty} \frac{\sqrt{9^x+2^x}}{\sqrt{9^x}}$$

$$= \lim_{x\to\infty} \sqrt{1+\left(\frac{2}{9}\right)^x} = 1$$

Thus f_1 to f_2 grow at the same rate.

Compare f_1 to f_3.

$$\lim_{x\to\infty} \frac{f_3(x)}{f_1(x)} = \lim_{x\to\infty} \frac{\sqrt{9^x-4^x}}{3^x}$$

$$= \lim_{x\to\infty} \frac{\sqrt{9^x-4^x}}{\sqrt{9^x}}$$

$$= \lim_{x\to\infty} \sqrt{1-\left(\frac{4}{9}\right)^x} = 1$$

Thus f_1 and f_3 grow at the same rate.

By transitivity, f_2 and f_3 grow at the same rate, so all three functions grow at the same rate as $x \to \infty$.

34. Compare f_1 to f_2.

$$\lim_{x\to\infty} \frac{f_2(x)}{f_1(x)} = \lim_{x\to\infty} \frac{\frac{x^4+2x^2-1}{x+1}}{x^3}$$

$$= \lim_{x\to\infty} \frac{x^4+2x^2-1}{x^4+x^3}$$

$$= \lim_{x\to\infty} \frac{1+\frac{2}{x^2}-\frac{1}{x^4}}{1+\frac{1}{x}} = 1$$

Thus f_1 and f_2 grow at the same rate.

Compare f_1 and f_3.

$$\lim_{x\to\infty} \frac{f_3(x)}{f_1(x)} = \lim_{x\to\infty} \frac{\frac{2x^5-1}{x^2+1}}{x^3}$$

$$= \lim_{x\to\infty} \frac{2x^5-1}{x^5+x^3}$$

$$= \lim_{x\to\infty} \frac{2-\frac{1}{x^5}}{1+\frac{1}{x^2}} = 2$$

Thus f_1 and f_3 grow at the same rate.

By transitivity, f_2 and f_3 grow at the same rate, so all three functions grow at the same rate.

35. f grows faster than g.

36. g grows faster than f.

37. f and g grow at the same rate.

38. f and g grow at the same rate.

39. (a) The nth derivation of x^n is $n!$, a constant. We can apply L'Hôpital's Rule n times to find $\lim_{x\to\infty} \frac{e^x}{x^n}$.

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \cdots = \lim_{x\to\infty} \frac{e^x}{n!} = \infty$$

Thus e^x grows faster than x^n as $x \to \infty$ for any positive integer n.

(b) The nth derivative of a^x. $a > 1$, is $(\ln a)^n a^x$. We can apply L' Hôpital's Rule n times to find $\lim_{x\to\infty} \frac{a^x}{x^n}$.

$$\lim_{x\to\infty} \frac{a^x}{x^n} = \cdots = \lim_{x\to\infty} \frac{(\ln a)^n a^x}{n!} = \infty$$

39. Continued

(b) Thus a^x grows faster than x^n as $x \to \infty$ for any positive integer *n*.

40. (a) Apply L'Hôpital's Rule *n* times to find

$$\lim_{x\to\infty} \frac{e^x}{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}.$$

$$\lim_{x\to\infty} \frac{e^x}{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0} = \lim_{x\to\infty} \frac{e^x}{a_n n!} = \infty$$

Thus e^x grows faster than

$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ as $x \to \infty$.

(b) Apply L'Hôpital's Rule *n* times to find

$$\lim_{x\to\infty} \frac{a^x}{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}.$$

$$\lim_{x\to\infty} \frac{a^x}{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0} = \cdots$$

$$= \lim_{x\to\infty} \frac{(\ln a)^n a^x}{a_n n!} = \infty$$

Thus a_x grows faster than

$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ as $x \to \infty$.

41. (a) $\lim_{x\to\infty} \frac{\ln x}{x^{1/n}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{n} x^{(1/n)-1}} = \lim_{x\to\infty} \frac{n}{x^{1/n}} = 0$

Thus ln *x* grows slower than $x^{1/n}$ as $x \to \infty$ for any positive integer *n*.

(b) $\lim_{x\to\infty} \frac{\ln x}{x^a} = \lim_{x\to\infty} \frac{\frac{1}{x}}{ax^{a-1}} = \lim_{x\to\infty} \frac{1}{ax^a} = 0$

Thus ln *x* grows slower than x^a as $x \to \infty$ for any number a > 0.

42. $\lim_{x\to\infty} \frac{\ln x}{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1}$$

$$= \lim_{x\to\infty} \frac{1}{na_n x^n + (n-1)a_{n-1}x^{n-1} + \cdots + a_1 x} = 0$$

Thus ln *x* grows than any nonconstant polynomial as to $x \to \infty$.

43. Compare $n \log_2 n$ to $n^{3/2}$ as $n \to \infty$.

$$\lim_{n\to\infty} \frac{n \log_2 n}{n^{3/2}} = \lim_{n\to\infty} \frac{\log_2 n}{n^{1/2}} = \lim_{n\to\infty} \frac{\frac{(\ln n)}{(\ln 2)}}{n^{1/2}} = \lim_{n\to\infty} \frac{\frac{1}{n \ln 2}}{\frac{1}{2n^{1/2}}} = \lim_{n\to\infty} \frac{2}{n^{1/2}(\ln 2)} = 0$$

Thus $n \log_2 n$ grows slower than $n^{3/2}$ as $n \to \infty$.

Compare $n \log_2 n$ to $n(\log_2 n)^2$

$$\lim_{n\to\infty} \frac{n \log_2 n}{n(\log_2 n)^2} = \lim_{n\to\infty} \frac{1}{\log_2 n} = 0$$

Thus $n \log_2 n$ grows slower than $n(\log_2 n)^2$ as $n \to \infty$.

The algorithm of order of $n \log_2 n$ is likely the most efficient because of the three functions, it grows the most slowly as $n \to \infty$.

44. (a) It might take 1,000,000 searches if it is the last item in the search.

(b) $\log_2 1{,}000{,}000 \approx 19.9$; it might take 20 binary searches.

45. (a) The limit will be the ratio of the leading coefficients of the polynomials since the polynomials must have the same degree.

(b) By the same reason as in (a), the limit will be the ratio of the leading coefficients of the polynomial.

46. True. because $\lim_{n\to\infty} \frac{n \log_2 n}{n^{3/2}} = 0$

47. False. They grow at the same rate.

48. E. $\lim_{x\to\infty} \frac{x^6+1}{x^5+x^2+1} = \lim_{x\to\infty} \frac{6!x}{5!} = \infty$

49. A. $\lim_{x\to\infty} \frac{\log_3 x}{e^{-x}} = \frac{1/x \ln 13}{-e^{-x}} = 0$

50. C. $\lim_{x\to\infty} \frac{e^{x+2}}{e^x} = e^2$

51. D. $\lim_{x\to\infty} \frac{\sqrt{x^8+x^4}}{x^4} = \lim_{x\to\infty} \frac{x^8+x^4}{x^8} = \lim_{x\to\infty} \frac{6720x^3}{6720x^3} = 1$

52. (a) $\lim_{x\to\infty} \frac{x^5}{x^2} = \lim_{x\to\infty} x^3 = \infty$

x^5 grows faster than x^2.

(b) $\lim_{x\to\infty} \frac{5x^3}{2x^3} = \lim_{x\to\infty} \frac{5}{2} = \frac{5}{2}$

$5x^3$ and $2x^3$ have the same rate of growth.

52. Continued

(c) $m > n$ since $\lim_{x\to\infty} \frac{x^m}{x^n} = \lim_{x\to\infty} x^{m-n} = \infty$.

(d) $m = n$ since $\lim_{x\to\infty} \frac{x^m}{x^n} = \lim_{x\to\infty} x^{m-n}$ is nonzero and finite.

(e) Degree of g > degree of f ($m > n$) since $\lim_{x\to\infty} \frac{g(x)}{f(x)} = \infty$.

(f) Degree of g = degree of f ($m = n$) since $\lim_{x\to\infty} \frac{g(x)}{f(x)}$ is nonzero and finite.

53. (a) $\lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} = \lim_{x\to\infty} \frac{-f(x)}{-g(x)} = \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$

Thus $|f|$ grows faster than $|g|$ as $x \to \infty$ by definition.

(b) $\lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} = \lim_{x\to\infty} \frac{-f(x)}{-g(x)} = \lim_{x\to\infty} \frac{f(x)}{g(x)} = L$

Thus $|f|$ grows at the same rate as $|g|$ as $x \to \infty$ by definition.

54. (a) $\lim_{x\to\infty} \frac{f(-x)}{g(-x)} = \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$

Thus $f(-x)$ grows faster than $g(-x)$ by definition.

(b) $\lim_{x\to\infty} \frac{f(-x)}{g(-x)} = \lim_{x\to\infty} \frac{f(x)}{g(x)} = L$

Thus $f(-x)$ grows at the same rate as $g(-x)$ by definition.

Section 8.4 Improper Integrals (pp. 459–470)

Exploration 1 Investigating $\int_0^1 \frac{dx}{x^p}$

1. Because $\frac{1}{x^p}$ has an infinite discontinuity at $x = 0$.

2. $\int_0^1 \frac{dx}{x} = \lim_{c\to 0^+} \int_c^1 \frac{dx}{x} = \lim_{c\to 0^+} \left[\ln x\right]_c^1 = \lim_{c\to 0^+} (-\ln c) = \infty$

3. If $p>1$, then

$$\int_0^1 \frac{dx}{x^p} = \lim_{c\to 0^+} \int_c^1 \frac{dx}{x^p}$$

$$= \lim_{c\to 0^+} \left.\frac{x^{-p+1}}{-p+1}\right]_c^1$$

$$= \lim_{c\to 0^+} \left(\frac{1-c^{-p+1}}{-p+1}\right) = \infty \text{ because } (-p+1) < 0.$$

4. If $0 < p < 1$, then

$$\int_x^1 \frac{dx}{x^p} = \lim_{c\to 0^+} \int_c^1 \frac{dx}{x^p}$$

$$= \lim_{c\to 0^+} \left.\frac{x^{-p+1}}{-p+1}\right]_c^1$$

$$= \lim_{c\to 0^+} \left(\frac{1-c^{-p+1}}{-p+1}\right) = \frac{1}{1-p}$$

Quick Review 8.4

1. $\int_0^3 \frac{dx}{x+3} = \Big[\ln|x+3|\Big]_0^3 = \ln 6 - \ln 3 = \ln 2$

2. $\int_{-1}^1 \frac{x\,dx}{x^2+1} = \left[\frac{1}{2}\ln|x^2+1|\right]_{-1}^1 = \frac{1}{2}\ln 2 - \frac{1}{2}\ln 2 = 0$

3. $\int \frac{dx}{x^2+4} = \frac{1}{4}\int \frac{dx}{\left(\frac{x}{2}\right)^2+1}$

$$= \frac{1}{4}\left(2\tan^{-1}\frac{x}{2}\right) + C$$

$$= \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

4. $\int \frac{dx}{x^4} = \int x^{-4}\,dx = -\frac{1}{3}x^{-3} + C$

5. $9 - x^2 > 0$ for $-3 < x < 3$
The domain is $(-3, 3)$.

6. $x - 1 > 0$ for $x > 1$
The domain is $(1, \infty)$.

7. $-1 \le \cos x \le 1$, so $|\cos x| \le 1$.

$$\left|\frac{\cos x}{x^2}\right| = \frac{|\cos x|}{|x^2|} \le \frac{1}{x^2}$$

8. $x^2 - 1 \le x^2$ so $\sqrt{x^2-1} \le \sqrt{x^2} = x$ for $x > 1$

$$\frac{1}{\sqrt{x^2-1}} \ge \frac{1}{x}$$

9. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{4e^x - 5}{3e^x + 7} = \lim_{x\to\infty} \frac{4e^x}{3e^x} = \lim_{x\to\infty} \frac{4}{3} = \frac{4}{3}$

Thus f and g grow at the same rate as $x \to \infty$.

10. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\sqrt{2x-1}}{\sqrt{x+3}}$

$$= \lim_{x\to\infty} \frac{\sqrt{2x-1}}{x+3}$$

$$= \lim_{x\to\infty} \sqrt{\frac{2-\frac{1}{x}}{1+\frac{3}{x}}} = \sqrt{2}$$

Section 8.4 Exercises

1. (a) $\int_0^{\infty} \frac{2x}{x^2+1}dx = \lim_{b\to\infty}\int_0^b \frac{2x}{x^2+1}dx$

(b) $\lim_{b\to\infty}\int_0^b \frac{2x}{x^2+1}dx = \lim_{b\to\infty}\left(\ln(x^2+1)\right)\Big|_0^b = \infty$

diverges

2. (a) $\int_1^{\infty}\frac{dx}{x^{1/3}} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^{1/3}}$

(b) $\lim_{b\to\infty}\int_1^b \frac{dx}{x^{1/3}} = \lim_{b\to\infty}\left(\frac{3}{2}x^{2/3}\right)\Big|_1^b = \infty$

diverges

3. (a) $\int_{-\infty}^{\infty}\frac{2x}{(x^2+1)^2}dx = \lim_{b\to-\infty}\int_b^0 \frac{2x}{(x^2+1)^2}dx + \lim_{b\to\infty}\int_0^b \frac{2x}{(x^2+1)^2}dx$

(b) $\lim_{b\to-\infty}\int_b^0 \frac{2x}{(x^2+1)^2}dx + \lim_{b\to\infty}\int_0^b \frac{2x}{(x^2+1)^2}dx$

$= \lim_{b\to-\infty}\left(\frac{-1}{x^2+1}\right)\Big|_b^0 + \lim_{b\to\infty}\left(\frac{-1}{x^2+1}\right)\Big|_0^b = 0$

converges

4. (a) $\int_1^{\infty}\frac{dx}{\sqrt{x}} = \lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt{x}}$

(b) $\lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt{x}} = \lim_{b\to\infty}(2\sqrt{x})\Big|_1^b = \infty$

diverges

5. $\int_1^{\infty}\frac{dx}{x^4} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^4} = \lim_{b\to\infty}\left(\frac{-1}{3x^3}\right)\Big|_1^b = \frac{1}{3}$

6. $\int_1^{\infty}\frac{2dx}{x^3} = \lim_{b\to\infty}\int_1^b \frac{2dx}{x^3} = \lim_{b\to\infty}\left(\frac{-2}{2x^2}\right)\Big|_1^b = 1$

7. $\int_1^{\infty}\frac{dx}{\sqrt[3]{x}} = \lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt[3]{x}} = \lim_{b\to\infty}\left(\frac{3}{2}(x)^{2/3}\right)\Big|_1^b = \infty$

diverges

8. $\int_1^{\infty}\frac{dx}{\sqrt[4]{x}} = \lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt[4]{x}} = \lim_{b\to\infty}\left(\frac{4}{3}(x)^{3/4}\right)\Big|_1^b = \infty$

diverges

9. $\int_{-\infty}^{-1}\frac{dx}{x^2} = \lim_{b\to-\infty}\int_b^{-1}\frac{dx}{x^2} = \lim_{b\to\infty}\left(-\frac{1}{x}\right)\Big|_b^{-1} = 1$

10. $\int_{-\infty}^{0}\frac{dx}{(x-2)^3} = \lim_{b\to-\infty}\int_b^0 \frac{dx}{(x-2)^3} = \lim_{b\to\infty}\left(\frac{-1}{2(x-2)^2}\right)\Big|_b^0 = -1/8$

11. $\int_{-\infty}^{-2}\frac{2dx}{x^2-1} = \lim_{b\to-\infty}\int_b^{-2}\frac{2dx}{x^2-1} = \lim_{b\to-\infty}\left(-\ln\left(\frac{x+1}{x-1}\right)\right)\Big|_b^{-2} = \ln 3$

12. $\int_2^{\infty}\frac{3dx}{x^2-x} = \lim_{b\to\infty}\int_2^b \frac{3dx}{x^3-x} = \lim_{b\to\infty}\left(3\ln\left(\frac{x-1}{x}\right)\right)\Big|_2^b = 3\ln 2$

13. $\int_{-1}^{\infty}\frac{dx}{x^2+5x+6} = \lim_{b\to\infty}\int_{-1}^b \frac{dx}{x^2+5x+6} = \lim_{b\to\infty}\left(-\ln\left(\frac{x+3}{x+2}\right)\right)\Big|_{-1}^b$

$= \ln 2$

14. $\int_{-\infty}^{0}\frac{2dx}{x^2-4x+3} = \lim_{b\to-\infty}\int_b^0 \frac{2dx}{x^2-4x+3}$

$= \lim_{b\to-\infty}\left(-\ln\left(\frac{x-1}{x-3}\right)\right)\Big|_b^0 = \ln 3$

15. $\int_1^{\infty}\frac{5x+6}{x^2+2x}dx = \lim_{b\to\infty}\int_1^b \frac{5x+6}{x^2+2x}dx$

$= \lim_{b\to\infty}\left(\ln((x+2)^2(x)^3)\right)\Big|_1^b = \infty$

diverges

16. $\int_{-2}^{-\infty}\frac{2dx}{x^2-2x} = \lim_{b\to-\infty}\int_{-2}^b \frac{2dx}{x^2-2x}$

$= \lim_{b\to-\infty}\left(\ln\left(\frac{x-2}{x}\right)\right)\Big|_{-2}^b = -\ln 2$

17. $\int_1^{\infty} xe^{-2x}dx = \lim_{b\to\infty}\int_1^b xe^{-2x}dx = \lim_{b\to\infty}\left(\frac{-x}{2}-\frac{1}{4}\right)e^{-2x} = \frac{3}{4}e^{-2}$

18. $\int_{-\infty}^{0} x^2e^x dx = \lim_{b\to-\infty}\int_b^0 x^2e^x dx$

$= \lim_{b\to-\infty}\left(\left(\frac{-x^2}{2}-\frac{x}{2}-\frac{1}{4}\right)e^{-2x}\right)\Big|_b^0 = 2$

19. $\int_1^{\infty} x\ln x\,dx = \lim_{b\to\infty}\int_1^b x\ln x\,dx = \lim_{b\to\infty}\left(\frac{x^2}{2}\ln 2 - \frac{x^2}{4}\right)\Big|_1^b = \infty$

diverges

20. $\int_0^{\infty}(x+1)e^{-x}dx = \lim_{b\to\infty}\int_0^b (x+1)e^{-x}dx$

$= \lim_{b\to\infty}\left((-x-2)e^{-x}\right)\Big|_0^b = 2$

21. $\int_{-\infty}^{\infty} e^{-1x1}dx = \lim_{b\to-\infty}\int_b^0 e^{-1x1}dx + \lim_{b\to\infty}\int_0^b e^{-1x1}dx = 2$

22. $\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx = \lim_{b\to\infty}\int_b^0 2xe^{-x^2}\,dx + \lim_{b\to\infty}\int_0^b 2xe^{-x^2}\,dx =$

$\lim_{b\to-\infty}\left(-e^{-x^2}\right)\Big|_{-b}^{0} + \lim_{b\to\infty}\left(-e^{-x^2}\right)\Big|_0^b = 0$

23. $\int_{-\infty}^{\infty}\frac{dx}{e^x+e^{-x}} = \lim_{b\to-\infty}\int_{-b}^{0}\frac{dx}{e^x+e^{-x}} + \lim_{b\to\infty}\int_0^b\frac{dx}{e^x+e^{-x}}$

$\lim_{b\to-\infty}\left(\tan^{-1}(e^x)\right)\Big|_b^0 + \lim_{b\to\infty}\left(\tan^{-1}(e^x)\right)\Big|_0^b = \frac{\pi}{2}$

24. $\int_{-\infty}^{\infty} e^{2x}\,dx = \lim_{b\to-\infty}\int_b^0 e^{2x}\,dx + \lim_{b\to\infty}\int_0^b e^{2x}\,dx$

$= \lim_{b\to-\infty}\left(\frac{e^{2x}}{2}\right)\Big|_b^0 + \lim_{b\to\infty}\left(\frac{e^{2x}}{2}\right)\Big|_0^b = \infty$

diverges

25. **(a)** The integral has an infinite discontinuity at the interior point $x = 1$.

(b) $\int_0^2\frac{dx}{1-x^2} = \lim_{b\to1^-}\int_0^b\frac{dx}{1-x^2} + \lim_{b\to1^+}\int_b^2\frac{dx}{1-x^2}$

$= \lim_{b\to1^-}\left(\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)\right)\Big|_0^b + \lim_{b\to1^+}\left(\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)\right)\Big|_b^2 = \infty$

diverges

26. **(a)** The integral has an infinite discontinuity at the endpoint $x = 1$.

(b) $\int_0^1\frac{dx}{\sqrt{1-x^2}} = \lim_{b\to1}\int_0^b\frac{dx}{\sqrt{1-x^2}} = \lim_{b\to1}\left(\sin^{-1}(x)\right)\Big|_0^b = \frac{\pi}{2}$

27. **(a)** The integral has an infinite discontinuity at the endpoint $x = 0$.

(b) $\int_0^1\frac{x+1}{\sqrt{x^2+2x}}\,dx = \lim_{b\to0}\int_b^1\frac{x+1}{\sqrt{x^2+2x}}\,dx$

$= \lim_{b\to0}\left(\sqrt{x^2+2x}\right)\Big|_b^1 = \sqrt{3}$

28. **(a)** The integral has an infinite discontinuity at the endpoint $x = 0$.

(b) $\int_0^4\frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{b\to0}\int_b^4\frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{b\to0}\left(-2e^{-\sqrt{x}}\right)\Big|_b^4 = 2-2e^{-2}$

29. **(a)** The integral has an infinite discontinuity at the endpoint $x = 0$.

(b) $\int_0^1 x\ln(x)\,dx = \lim_{b\to0}\int_b^1 x\ln(x)\,dx$

$= \lim_{b\to0}\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\Big|_b^1 = -1/4$

30. **(a)** The integral has an infinite discontinuity at the interior point $x = 0$.

(b) $\int_{-1}^4\frac{dx}{\sqrt{|x|}} = \lim_{b\to0^-}\int_{-1}^b\frac{dx}{\sqrt{|x|}} + \lim_{b\to0^+}\int_b^4\frac{dx}{\sqrt{|x|}}$

$= \lim_{b\to0^-}\left(2\sqrt{x}\sqrt{\operatorname{sign}(x)}\right)\Big|_{-1}^b +$

$\lim_{b\to0^+}\left(2\sqrt{x}\sqrt{\operatorname{sign}(x)}\right)\Big|_b^4 = 6$

31. $0 \le \frac{1}{1+e^x} \le \frac{1}{e^x}$ on $[1,\infty)$, converges because $\int_1^\infty\frac{1}{e^x}dx$ converges.

32. $0 \le \frac{1}{x^3+1} \le \frac{1}{x^3}$ on $[1,\infty)$, converges because $\int_1^\infty\frac{1}{x^3}dx$ converges.

33. $0 \le \frac{1}{x} \le \frac{2+\cos x}{x}$ on $[\pi,\infty)$, diverges because $\int_\pi^\infty\frac{1}{x}dx$ diverges.

34. $\int_{-\infty}^{\infty}\frac{dx}{\sqrt{x^4+1}} = 2\int_0^\infty\frac{dx}{\sqrt{x^4+1}} = 2\int_0^1\frac{dx}{\sqrt{x^4+1}} + 2\int_0^\infty\frac{dx}{\sqrt{x^4+1}}$

and $0 \le \frac{1}{\sqrt{x^4+1}} \le \frac{1}{x^2}$ on $[1,\infty)$

converges because $\int_1^\infty\frac{1}{x^2}\,dx$ converges.

35. $\int_0^{\ln 2} y^{-2}e^{1/y}\,dy = \lim_{b\to0}\int_b^{\ln 2} y^{-2}e^{1/y}\,dy = \lim_{b\to0}\left(-e^{1/x}\right)\Big|_b^{\ln 2} = -\infty$

diverges

36. $\int_0^4\frac{dr}{\sqrt{4-r}} = \lim_{b\to4}\int_0^b\frac{dr}{\sqrt{4-r}} = \lim_{b\to4}\left(-2\sqrt{4-r}\right)\Big|_0^b = 4$

37. $\int_0^\infty\frac{ds}{(1+s)\sqrt{s}} = \lim_{b\to0}\int_b^\infty\frac{ds}{(1+s)\sqrt{s}} = \lim_{b\to0}\left(2\tan^{-1}\sqrt{s}\right)\Big|_b^\infty = \pi$

38. $\int_1^2\frac{du}{u\sqrt{u^2-1}} = \lim_{b\to1}\int_b^2\frac{du}{u\sqrt{u^2-1}} = \lim_{b\to1}\left(\tan^{-1}\sqrt{x^2-1}\right)\Big|_b^2 = \frac{\pi}{3}$

39. $\int_0^\infty\frac{16\tan^{-1}v}{1+v^2}dv = \lim_{b\to\infty}\int_0^b\frac{16\tan^{-1}v}{1+v^2}dv = \lim_{b\to\infty}\left(8(\tan^{-1}v)^2\right)\Big|_0^b$

$= 2\pi^2$

40. $\int_{-\infty}^0\theta e^\theta\,d\theta = \lim_{b\to-\infty}\int_b^0\theta e^\theta\,d\theta = \lim_{b\to-\infty}\left((x-1)e^x\right)\Big|_b^0 = -1$

41. $\int_0^2\frac{dt}{1-t} = \lim_{b\to1^-}\int_0^b\frac{dt}{1-t} + \lim_{b\to1^+}\int_b^2\frac{dt}{1-t} = \lim_{b\to1^-}\left(-\ln(x-1)\right)\Big|_0^b$

$+ \lim_{b\to1^+}\left(-\ln(x-1)\right)\Big|_b^2 = -\infty$

diverges

42. $\int_{-1}^1\ln(|w|)\,dw = \lim_{b\to0^-}\int_{-1}^b\ln(|w|)\,dw + \lim_{b\to0^+}\int_b^1\ln(|w|)\,dw$

$= \lim_{b\to0^-}\left(w\ln(|w|)-w\right)\Big|_{-1}^b + \lim_{b\to0^+}\left(w\ln(|w|)-w\right)\Big|_b^1 = -2$

43. For $x \geq 0, y \geq 0$ on $[1,\infty)$.

$$\text{Area} = \int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx$$

Integrate $\int \frac{\ln x}{x^2}\,dx$ by parts.

$$u = \ln x \qquad dv = \frac{dx}{x^2}$$

$$du = \frac{1}{x}dx \qquad v = -\frac{1}{x}$$

$$\int \frac{\ln x}{x^2} = -\frac{\ln x}{x} + \int \frac{dx}{x^2} = -\frac{\ln x}{x} - \frac{1}{x} + C$$

$$\text{Area} = \lim_{b\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right] = 1$$

$$\left(\text{Note that } \lim_{b\to\infty}\frac{\ln b}{b} = \lim_{b\to\infty}\frac{1/b}{1} = 0.\right)$$

44. For $x \geq 0$, $y \geq 0$ on $[1, \infty)$.

$$\text{Area} = \int_1^\infty \frac{\ln x}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x}\,dx$$

Integrate $\int \frac{\ln x}{x}\,dx$ by letting $u =$ so $du = \frac{dx}{x}$.

$$\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$$

$$\text{Area} = \lim_{b\to\infty}\left[\frac{1}{2}(\ln x^2)\right]_1^b = \lim_{b\to\infty}\frac{1}{2}(\ln b)^2 = \infty$$

45. **(a)** Since f is even, $f(-x) = f(x)$. Let $u = -x$, $du = -dx$.

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx$$
$$= \int_\infty^0 f(-u)(-1)\,du + \int_0^\infty f(x)\,dx$$
$$= \int_0^\infty f(x)du + \int_0^\infty f(x)\,dx$$
$$= 2\int_0^\infty f(x)\,dx$$

(b) Since f is odd, $f(-x). = -f(x)$. Let $u = -x$, $du = -dx$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx$$
$$= \int_{-\infty}^0 f(-u)(-1)\,du + \int_0^\infty f(x)\,dx$$
$$= -\int_0^\infty f(u)\,du + \int_0^\infty f(x)\,dx = 0$$

46. **(a)** $\int_0^\infty \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty}\int_0^b \frac{2x\,dx}{x^2+1}$

$$= \lim_{b\to\infty}\left[\ln(x^2+1)\right]_0^b$$
$$= \lim_{b\to\infty}\ln(b^2+1) = \infty$$

Thus the integral diverges.

(b) Both $\int_0^\infty \frac{2x\,dx}{x^2+1}$ and $\int_{-\infty}^0 \frac{2x\,dx}{x^2+1}$ must converge in order for $\int_{-\infty}^\infty \frac{2x\,dx}{x^2+1}$ to converge.

(c) $\lim_{b\to\infty}\int_{-b}^b \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty}\left[\ln(x^2+1)\right]_{-b}^b$

$$= \lim_{b\to\infty}\left[\ln(b^2+1) - \ln(b^2+1)\right]$$
$$= \lim_{b\to\infty} 0 = 0.$$

Note that $\frac{2x}{x^2+1}$ is an odd function so $\int_{-b}^b \frac{2x\,dx}{x^2+1} = 0$.

(d) Because the determination of convergence is not made using the method in part (c). In order for the integral to converge, there must be finite areas both directions (toward ∞ and toward –∞). In this case, there are infinite areas in both directions, but when one computes the intergral over an interval [–*b, b*], there is cancellation which gives 0 as the result.

47. By symmetry, find the perimeter of one side, say for $0 \leq x \leq 1, y \geq 0$.

$$y^{2/3} = 1 - x^{2/3}$$

$$y = (1 - x^{2/3})^{3/2}$$

$$\frac{dy}{dx} = \frac{3}{2}(1-x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = -x^{-1/3}(1-x^{2/3})^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = x^{-2/3}(1-x^{2/3}) = (x^{-2/3}-1)$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{x^{-2/3}} = x^{-1/3}$$

$$\int_0^1 x^{-1/3}\,dx = \lim_{b\to 0^+}\int_b^1 x^{-1/3}\,dx$$
$$= \lim_{b\to 0^+}\left[\frac{3}{2}x^{2/3}\right]_b^1$$
$$= \lim_{b\to 0^+}\left[\frac{3}{2} - \frac{3}{2}b^{2/3}\right] = \frac{3}{2}$$

Thus, the perimeter is $4\left(\frac{3}{2}\right) = 6$.

48. False. See Theorem 6.

49. True. See Theorem 6.

50. C. $\int_1^\infty \frac{dx}{x^{1.01}} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^{1.01}} = \lim_{b\to\infty}\left(-\frac{100}{x^{0.01}}\right)\Bigg|_1^b = 100$

51. B. $\int_0^1 \frac{dx}{x^{0.5}} = \lim_{b\to 0}\int_b^1 \frac{dx}{x^{0.5}} = \lim_{b\to 0}\left(2\sqrt{x}\right)\Big|_b^1 = 2$

52. E. $\int_0^1 \frac{dx}{x-1} = \lim_{b\to 1}\int_0^b \frac{dx}{x-1} = \lim_{b\to 1}\left(\ln(|x-1|)\right)\Big|_0^b = -\infty$

53. C. $\int_0^\infty \frac{dx}{x^2+1} = \lim_{b\to\infty}\int_0^b \frac{dx}{x^2+1} = \lim_{b\to\infty}\left(\tan^{-1}x\right)\Big|_0^b = \frac{\pi}{2}$

54. (a) $\int_1^\infty \frac{dx}{x^{0.5}} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^{0.5}} = \lim_{b\to\infty}\left(2\sqrt{x}\right)\Big|_1^b = \infty$, or it diverges.

(b) $\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty}\int_1^b \frac{dx}{x} = \lim_{b\to\infty}\left(\ln|x|\right)\Big|_1^b = \infty$, or it diverges.

(c) $\int_1^\infty \frac{dx}{x^{1.5}} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^{1.5}} = \lim_{b\to\infty}\left(\frac{-2}{\sqrt{x}}\right)\Big|_1^b = 2$

(d) $\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^p}$

$= \lim_{b\to\infty}\left(\frac{1}{p-1}\frac{1}{x^{p-1}}\right)\Big|_1^b$

$= \lim_{b\to\infty}\left(\frac{1}{p-1}\frac{1}{b^{p-1}} - \frac{1}{p-1}\frac{1}{1^{p-1}}\right)$

$= \lim_{b\to\infty}\left(\frac{1}{1-p}\left(\frac{1}{b^{p-1}}-1\right)\right)$

(e) $\int_1^\infty \frac{dx}{x^p},\ p>1$

$= \lim_{b\to\infty}\left(\frac{1}{1-p}\frac{1}{b^{p-1}} - \frac{1}{1-p}\right)$

$= \frac{1}{1-p}\frac{1}{\infty} - \frac{1}{1-p}$

$= \frac{1}{p-1}$

$\int_1^\infty \frac{dx}{x^p},\ p<1$

$= \frac{1}{1-p}\frac{1}{b^{p-1}} - \frac{1}{1-p}$

$= \infty\frac{1}{b^{p-1}} - \infty$

$= \infty$

55. (a) $A(x) = \frac{\pi}{4}e^{2x}$

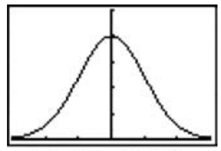
(b) $V = \int_{-\infty}^{\ln 2} A(x)\,dx = \int_{-\infty}^{\ln 2}\frac{\pi}{4}e^{2x}\,dx$

(c) $V = \int_{-\infty}^{\ln 2}\frac{\pi}{4}e^{2x}\,dx = \lim_{b\to-\infty}\int_b^{\ln 2}\frac{\pi}{4}e^{2x}\,dx$

$= \lim_{b\to-\infty}\left(\frac{\pi e^{2x}}{8}\right)\Bigg|_b^{\ln 2}$

$= \frac{\pi 4}{8} - 0 = \frac{\pi}{2}$

56. (a) $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$



[−3, 3] by [0, 0.5]

f is increasing on $(-\infty, 0]$, f is decreasing on $[0, \infty)$.

f has a local maximum at $(0, f(0)) = \left(0, \frac{1}{\sqrt{2\pi}}\right)$

(b) $\text{NINT}\left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}, x, -1, 1\right) \approx 0.683$

$\text{NINT}\left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}, x, -2, 2\right) \approx 0.954$

$\text{NINT}\left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}, x, -3, 3\right) \approx 0.997$

(c) Part (b) suggests that as b increases, the integral approaches 1. We can make $\int_{-b}^{b} f(x)\,dx$ as close to 1 as we want by choosing b > 1 large enough. Also we can make $\int_{-b}^{\infty} f(x)\,dx$ and $\int_{\infty}^{-b} f(x)\,dx$ as small as we want by choosing b large enough. This is because $0 < f(x) < e^{-x/2}$ for $x > 1$. (Likewise, $0 < f(x) < e^{x/2}$ for $x < -1$) Thus,

$\int_b^\infty f(x)\,dx < \int_b^\infty e^{-x/2}\,dx$

$\int_b^\infty e^{-x/2}\,dx = \lim_{c\to\infty}\int_b^c e^{-x/2}\,dx$

$= \lim_{c\to\infty}\left[-2e^{-x/2}\right]_b^c$

$= \lim_{c\to\infty}[-2e^{-c/2} + 2e^{-b/2}]$

$= 2e^{-b/2}$

As $b\to\infty$, $2e^{-b/2}\to 0$, so for large enough b, $\int_b^\infty f(x)\,dx$ is as small as we want. Likewise, for large enough b, $\int_{-\infty}^{-b} f(x)\,dx$ is as small as we want.

57. (a) For $x \ge 6, x^2 \ge 6x$, so $e^{-x^2} \le e^{-6x}$

$$\int_6^\infty e^{-x^2} dx \le \int_6^\infty e^{-6x}\,dx$$
$$= \lim_{b\to\infty} \int_6^b e^{-6x}\,dx$$
$$= \lim_{b\to\infty} \left[-\frac{1}{6}e^{-6x}\right]_6^b$$
$$= \lim_{b\to\infty} \left(-\frac{1}{6}e^{-6b} + \frac{1}{6}e^{-36}\right)$$
$$= \frac{1}{6}e^{-36} < 4\times 10^{-17}$$

(b) $$\int_1^\infty e^{-x^2}\,dx = \int_1^6 e^{-x^2}\,dx + \int_6^\infty e^{-x^2}\,dx$$
$$\le \int_1^6 e^{-x^2}\,dx + 4\times 10^{-17}$$

Thus, from part (a) we have shown that the error is bounded by 4×10^{-17}.

(c) $\int_1^\infty e^{-x^2}\,dx \approx \text{NINT}(e^{-x^2}, x, 1.6) \approx 0.1394027926$

(This agrees with Figure 8.16.)

(d) $$\int_0^\infty e^{-x^2}\,dx = \int_0^3 e^{x^2}\,dx + \int_3^\infty e^{x^2}\,dx$$
$$\le \int_0^3 e^{-x^2}\,dx + \int_3^\infty e^{-3x}\,dx$$

since $x^2 \ge 3x$ for $x > 3$.

$$\int_3^\infty e^{-3x}\,dx = \lim_{b\to\infty}\int_3^b e^{-3x}\,dx$$
$$= \lim_{b\to\infty}\left[-\frac{1}{3}e^{-3x}\right]_3^b$$
$$= \lim_{b\to\infty}\left(-\frac{1}{3}e^{-3b} + \frac{1}{3}e^{-9}\right)$$
$$= \frac{1}{3}e^{-9} \approx 0.000041 < 0.000042$$

58. Suppose $0 \le f(x) \le g(x)$ for all $x \ge a$.
From the properties of integrals, for any $b > a$,
$\int_a^b f(x) \le \int_a^b g(x)\,dx.$

If the infinite integral of g converges, then taking the limit in the above inequality as $b \to \infty$ shows that the infinite integral of f is bounded above by the infinite integral integral of g. Therefore, the infinite integral of f must be finite and it converges. If the infinite integral of f diverges, it must grow to infinity. So taking the limit in the above inequality as $b \to \infty$ shows that the infinite integral of g must also diverage to infinity.

59. (a) For $n = 0$:

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx$$
$$= \lim_{b\to\infty}\left[-e^{-x}\right]_0^b$$
$$= \lim_{b\to\infty}[-e^{-b}+1] = 1$$

For $n = 1$:

$u = x \qquad dv = e^{-x}\,dx$

$du = dx \qquad v = -e^{-x}$

$$\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx$$
$$= \lim_{b\to\infty}\left(\left[-xe^{-x}\right]_0^b + \int_0^b e^{-x}\,dx\right)$$
$$= \lim_{b\to\infty}\left(-\frac{b}{e^b}\right) + \lim_{b\to\infty}\int_0^b e^{-x}\,dx$$
$$= \lim_{b\to\infty}\left(-\frac{1}{e^b}\right) + 1 = 1$$

For $n = 2$;

$u = x^2 \qquad dv = e^{-x}\,dx$

$du = 2x\,dx \qquad v = -e^{-x}$

$$\int_0^\infty x^2 e^{-x}\,dx = \lim_{b\to\infty}\int_0^b x^2 e^{-x}\,dx$$
$$= \lim_{b\to\infty}\left(\left[-x^2e^{-x}\right]_0^b + \int_0^b 2x\,e^{-x}\,dx\right)$$
$$= \lim_{b\to\infty}\left(-\frac{b^2}{e^b}\right) + 2\lim_{b\to\infty}\int_0^b xe^{-x}\,dx$$
$$= \lim_{b\to\infty}\left(-\frac{2b}{e^b}\right) + 2(1)$$
$$= \lim_{b\to\infty}\left(-\frac{2b}{e^b}\right) + 2 = 2$$

(b) Evaluate $\int x^n e^{-x}\,dx$ using integration by parts

$u = x^n \qquad dv = e^{-x}\,dx$

$du = nx^{n-1} \qquad v = -e^{-x}$

$$\int x^n e^{-x}\,dx = -x^n e^{-x} + \int nx^{n-1} e^{-x}\,dx$$

$$f(n+1) = \int_0^\infty x^n e^{-x}\,dx$$
$$= \lim_{b\to\infty}\left[-x^n e^{-x}\right]_0^b + \int_0^\infty nx^{n-1}e^{-x}\,dx$$
$$= \lim_{b\to\infty}\left(-\frac{b^n}{e^b}\right) + n\int_0^\infty x^{n-1}e^{-x}\,dx$$
$$= nf(n)$$

$\Big($Note: apply L'Hôpital's Rule n times to show that $\lim_{b\to\infty}\left(-\frac{b^n}{e^b}\right) = 0.\Big)$

59. Continued

(c) Since $f(n+1) = nf(n)$,

$f(n+1) = n(n-1)\cdots f(1) = n!$; thus

$\int_0^{\infty} x^n e^{-x}\,dx$ converges for all integers $n \ge 0$.

60. (a) On a grapher, plot $\text{NINT}\left(\frac{\sin x}{x}, x, 0, x\right)$ or create a table of values. For large values of x, $f(x)$ appears to approach approximately 1.57.

(b) Yes, the integral appears to converge.

61. (a) $$\int_{-\infty}^{1} \frac{dx}{1+x^2} = \lim_{b\to-\infty} \int_b^1 \frac{dx}{1+x^2}$$
$$= \lim_{b\to-\infty} \left[\tan^{-1} x\right]_b^1$$
$$= \lim_{b\to-\infty} (\tan^{-1} 1 - \tan^{-1} b)$$
$$= \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$$
$$\int_1^{\infty} \frac{dx}{1+x^2} = \lim_{x\to\infty} \int_1^b \frac{dx}{1+x^2}$$
$$= \lim_{b\to\infty} \left[\tan^{-1} x\right]_1^b$$
$$= \lim_{b\to\infty} [\tan^{-1} b - \tan^{-1} 1]$$
$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{3\pi}{4} + \frac{\pi}{4} = \pi$$

(b) $$\int_{-\infty}^{c} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^c f(x)\,dx$$
$$\int_c^{\infty} f(x)\,dx = \int_c^0 f(x)\,dx + \int_0^{\infty} f(x)\,dx$$
Thus,
$$\int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$
$$= \int_{-\infty}^{0} f(x)\,dx + \int_0^c f(x)\,dx + \int_c^0 f(x)\,dx + \int_0^{\infty} f(x)\,dx$$
$$= \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx, \text{ because}$$
$$\int_c^c f(x)\,dx + \int_c^0 f(x)\,dx = \int_0^c f(x)\,dx - \int_0^c f(x)\,dx = 0.$$

Quick Quiz Sections 8.3 and 8.4

1. E. $\lim_{x\to\infty} \frac{0.1x^3}{x^2} = \lim_{x\to\infty} \frac{0.3x^2}{2x} = \lim_{x\to\infty} \frac{0.6x}{2} = \infty.$

2. C. Since $\int_1^{\infty} \frac{dx}{x^p} = \frac{1}{p-1}$ for $p > 1$, $p > 0$.

3. B. $\int_0^1 \frac{dx}{x^{p+1}} = C$ when $p < 0$.

4. (a) $\int_1^b \frac{2\ln(x)}{x^2}\,dx$

(b) $\lim_{b\to\infty} \int_1^{\infty} \frac{2\ln x}{x^2}\,dx$

(c) Note that $\int \frac{2\ln x}{x^2}\,dx$ can be found by parts.

Let $u = 2\ln x$ and $dv = x^{-2}\,dx$.

Then
$$\int \frac{2\ln x}{x^2}\,dx = 2\ln x(-x^{-1}) - \int (-x^{-1})2x^{-1}\,dx$$
$$= -\frac{2\ln x}{x} + \int \frac{2}{x^2}\,dx$$
$$= -\frac{2\ln x}{x} - \frac{2}{x} + C$$
$$\text{Area} = \lim_{b\to\infty} \int_1^b \frac{2\ln x}{x^2}\,dx$$
$$= \lim_{b\to\infty} \left[-\frac{2\ln x}{x} - \frac{2}{x}\right]_1^b$$
$$= \lim_{b\to\infty} \left(-\frac{2\ln b}{b} - \frac{2}{b} + 0 + 2\right) = 2$$

Chapter 8 Review Exercises (pp. 470–471)

1. $-1/2,\ 3/5, -2/3,\ 5/7$

$a_{40} = -(-1)^{40} \frac{40+1}{40+3} = 41/43$

2. $-3, -6, -12, -24$

$a_{40} = -3(2^{39})$

3. (a) $1/2 - (-1) = 3/2$

(b) $-1 + 9(3/2) = 25/2$

(c) $a_n = -1 + (n-1)3/2 = \frac{3n-5}{2}$

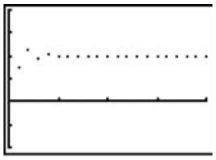
4. (a) $\frac{-2}{1/2} = -4$

(b) $\frac{1}{2}(-4)^6 = 2048$

(c) $a_n = (-1)^{n-1}(4)^n\left(\frac{1}{2}\right)$

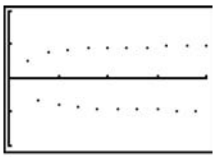
$= (-1)^{n-1}(2^{2n-3})$

5.



[0, 20] by [–2, 4]

6.



[0, 20] by [–2, 2]

7. $a_n = \lim\limits_{n\to\infty}\dfrac{3n^2-1}{2n^2+1} = \lim\limits_{n\to\infty}\dfrac{6n}{4n} = \lim\limits_{n\to\infty}\dfrac{3}{2} = \dfrac{3}{2}$, it converges.

8. $a_n = (-1)^n\dfrac{3n-1}{n+2},\ n = 2k, \lim\limits_{n\to\infty}(-1)^n\dfrac{3n-1}{n+2} > 0.$

$a_n = (-1)^n\dfrac{3n-1}{n+2},\ n = 2k^{-1}, \lim\limits_{n\to\infty}(-1)^n\dfrac{3n-1}{n+2} < 0.$

diverges.

9. $\lim\limits_{t\to0}\dfrac{t-\ln(1+2t)}{t^2} = \lim\limits_{t\to0}\dfrac{1-\dfrac{2}{1+2t}}{2t} = \infty$ for $t\to0^-$ and $-\infty$ for $t\to0^+$

The limit does not exit.

10. $\lim\limits_{t\to0}\dfrac{\tan 3t}{\tan 5t} = \lim\limits_{t\to0}\dfrac{3\sec^2 3t}{5\sec^2 5t} = \dfrac{3}{5}$

11. $\lim\limits_{x\to0}\dfrac{x\sin x}{1-\cos x} = \lim\limits_{x\to0}\dfrac{x\cos x+\sin x}{\sin x}$

$= \lim\limits_{x\to0}\dfrac{-x\sin x+\cos x+\cos x}{\cos x} = 2$

12. The limit leads to the indeterminate form 1^∞.

$f(x) = x^{1/(1-x)}$

$\ln f(x) = \dfrac{\ln x}{1-x}$

$\lim\limits_{x\to1}\dfrac{\ln x}{1-x} = \lim\limits_{x\to1}\dfrac{1/x}{-1} = -1$

$\lim\limits_{x\to1} x^{1/(1-x)} = \lim\limits_{x\to1} e^{\ln f(x)} = e^{-1} = \dfrac{1}{e}$

13. The limit leads to the indeterminate form ∞^0.

$f(x) = x^{1/x}$

$\ln f(x) = \dfrac{\ln x}{x}$

$\lim\limits_{x\to\infty}\dfrac{\ln x}{x} = \lim\limits_{x\to\infty}\dfrac{1/x}{1} = 0$

$\lim\limits_{x\to\infty} x^{1/x} = \lim\limits_{x\to\infty} e^{\ln f(x)} = e^0 = 1$

14. The limit leads to the indeterminate form 1^∞.

$f(x) = \left(1+\dfrac{3}{x}\right)^x$

$\ln f(x) = x\ln\left(1+\dfrac{3}{x}\right) = \dfrac{\ln\left(1+\dfrac{3}{x}\right)}{\dfrac{1}{x}}$

$\lim\limits_{x\to\infty}\dfrac{\ln\left(1+\dfrac{3}{x}\right)}{\dfrac{1}{x}} = \lim\limits_{x\to\infty}\dfrac{\dfrac{-3/x^2}{1+3/x}}{-\dfrac{1}{x^2}} = \lim\limits_{x\to\infty}\dfrac{3x}{x+3} = 3$

$\lim\limits_{x\to\infty}\left(1+\dfrac{3}{x}\right)^x = \lim\limits_{x\to\infty} e^{\ln f(x)} = e^3$

15. $\lim\limits_{r\to\infty}\dfrac{\cos r}{\ln r} = 0$ since $|\cos r| \le 1$ and $\ln r\to\infty$ as $r\to\infty$.

16. $\lim\limits_{\theta\to\pi/2}\left(\theta-\dfrac{\pi}{2}\right)\sec\theta = \lim\limits_{\theta\to\pi/2}\dfrac{\theta-\dfrac{\pi}{2}}{\cos\theta} = \lim\limits_{\theta\to\pi/2}\dfrac{1}{-\sin\theta} = -1$

17. $\lim\limits_{x\to1}\left(\dfrac{1}{x-1}-\dfrac{1}{\ln x}\right) = \lim\limits_{x\to1}\left[\dfrac{\ln x - x+1}{(x-1)\ln x}\right]$

$= \lim\limits_{x\to1}\dfrac{\dfrac{1}{x}-1}{\dfrac{x-1}{x}+\ln x}$

$= \lim\limits_{x\to1}\dfrac{1-x}{x-1+x\ln x}$

$= \lim\limits_{x\to1}\dfrac{-1}{1+x/x+\ln x} = -\dfrac{1}{2}$

18. The limit leads to the indeterminate from ∞^0.

$f(x) = \left(1 = \dfrac{1}{x}\right)^x$

$\ln f(x) = x\ln\left(1+\dfrac{1}{x}\right) = \dfrac{\ln(1+1/x)}{1/x}$

$\lim\limits_{x\to0^+}\dfrac{\ln(1+1/x)}{1/x} = \lim\limits_{x\to0^+}\dfrac{\dfrac{-1/x^2}{1+1/x}}{-1/x^2} = \lim\limits_{x\to0^+}\dfrac{x}{x+1} = 0$

$\lim\limits_{x\to0^+}\left(1+\dfrac{1}{x}\right)^x = \lim\limits_{x\to0^+} e^{\ln f(x) = e^0 = 1}$

19. The limit leads to the indeterminate from 0^0.

$f(\theta) = (\tan\theta)^\theta$

$\ln f(\theta) = \theta \ln(\tan\theta) = \dfrac{\ln(\tan\theta)}{1/\theta}$

$$\lim_{x\to 0^+} \frac{\ln(\tan\theta)}{1/\theta} = \lim_{x\to 0^+} \frac{\frac{\sec^2\theta}{\tan\theta}}{-\frac{1}{\theta^2}}$$
$$= \lim_{x\to 0^+} -\frac{\theta^2}{\sin\theta\cos\theta}$$
$$= \lim_{x\to 0^+} \frac{-2\theta}{-\sin^2\theta + \cos^2\theta} = 0$$

$$\lim_{x\to 0^+} (\tan\theta)^\theta = \lim_{x\to 0^+} e^{\ln f(\theta)} = e^0 = 1$$

20. $\displaystyle \lim_{\theta\to\infty} \theta^2 \sin\left(\frac{1}{\theta}\right) = \lim_{t\to 0^+} \frac{\sin t}{t^2} = \lim_{t\to 0^+} \frac{\cos t}{2t} = \infty$

21. $\displaystyle \lim_{x\to\infty} \frac{x^3 - 3x^2 + 1}{2x^2 + x - 3} = \lim_{x\to\infty} \frac{3x^2 - 6x}{4x + 1} = \lim_{x\to\infty} \frac{6x - 6}{4} = \infty$

22. $\displaystyle \lim_{x\to\infty} \frac{3x^2 - x + 1}{x^4 - x^3 + 2} = \lim_{x\to\infty} \frac{6x - 1}{4x^3 - 3x^2} = \lim_{x\to\infty} \frac{6}{12x^2 - 6x} = 0.$

23. $\displaystyle \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x}{5x} = \frac{1}{5}$

f grows at the same rate as *g*.

24. $\displaystyle \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\log_2 x}{\log_3 x} = \lim_{x\to\infty} \frac{(\ln x)/(\ln 2)}{(\ln x)/(\ln 3)} = \frac{\ln 3}{\ln 2}$

f grows at the same rate as *g*.

25. $\displaystyle \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x}{x + 1/x} = \lim_{x\to\infty} \frac{1}{1 - 1/x^2} = 1$

f grows at the same rate as *g*.

26. $\displaystyle \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x/100}{xe^{-x}} = \lim_{x\to\infty} \frac{e^x}{100} = \infty$

f grows faster than *g*.

27. $\displaystyle \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x}{\tan^{-1} x} = \infty$ since

$\displaystyle \lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$ and $\displaystyle \lim_{x\to\infty} x = \infty$

f grows faster than *g*.

28. $\displaystyle \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\csc^{-1} x}{1/x}$
$$= \lim_{x\to\infty} \frac{-\frac{1}{x\sqrt{x^2-1}}}{-\frac{1}{x^2}}$$
$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}}$$
$$= \lim_{x\to\infty} \sqrt{\frac{x^2}{x^2-1}}$$
$$= \lim_{x\to\infty} \sqrt{\frac{1}{1-1/x^2}} = 1$$

f grows at the same rate as *g*.

29. $\displaystyle \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{\ln x}}{x^{\log_2 x}}$
$$= \lim_{x\to\infty} x^{\ln x - \log_2 x}$$
$$= \lim_{x\to\infty} x^{\ln x - (\ln x)/\ln 2}$$
$$= \lim_{x\to\infty} x^{(\ln x)(1 - 1/\ln 2)}$$
$$= \lim_{x\to\infty} \left(\frac{1}{x}\right)^{(\ln x)(1/\ln 2 - 1)} = 0$$

Note that $1 - \dfrac{1}{\ln 2} < 0$ since $\ln 2 < 1$.

f grows slower than *g*.

30. $\displaystyle \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{3^{-x}}{2^{-x}} = \lim_{x\to\infty} \frac{2^x}{3^x} = \lim_{x\to\infty} \left(\frac{2}{3}\right)^x$

$= 0$ since $\dfrac{2}{3} < 1$.

f grows slower than *g*.

31. $\displaystyle \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\ln 2x}{\ln x^2} = \lim_{x\to\infty} \frac{\ln x + \ln 2}{2\ln x} = \lim_{x\to\infty} \frac{1/x}{2/x} = \frac{1}{2}$

f grows at the same rate as *g*.

32. $\displaystyle \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{10x^3 + 2x^2}{e^x}$
$$= \lim_{x\to\infty} \frac{30x^2 + 4x}{e^x}$$
$$= \lim_{x\to\infty} \frac{60x + 4}{e^x}$$
$$= \lim_{x\to\infty} \frac{60}{e^x} = 0$$

f grows slower than *g*.

33. $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{\tan^{-1}(1/x)}{1/x}$

$$=\lim_{x\to\infty}\frac{\frac{1}{1+(1/x)^2}(-x^{-2})}{-x^{-2}}$$

$$=\lim_{x\to\infty}\frac{1}{1+(1/x)^2}=1$$

f grows at the same rate as g.

34. $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{\sin^{-1}(1/x)}{(1/x^2)}$

$$=\lim_{x\to\infty}\frac{\frac{1}{\sqrt{1-(1/x)^2}}(-x^{-2})}{-2x^{-3}}$$

$$=\lim_{x\to\infty}\frac{x}{2\sqrt{1-(1/x)^2}}=\infty$$

f grows faster than g.

35. (a) $\lim_{x\to 0}f(x)=\lim_{x\to 0}\frac{2^{\sin x}-1}{e^x-1}$

$$=\lim_{x\to 0}\frac{(\ln 2)(\cos x)2^{\sin x}}{e^x}=\ln 2$$

(b) Define $f(0)=\ln 2$.

36. (a) $\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}x\ln x$

$$=\lim_{x\to 0^+}\frac{\ln x}{1/x}$$

$$=\lim_{x\to 0^+}\frac{1/x}{-1/x^2}$$

$$=\lim_{x\to 0^+}(-x)=0$$

(b) Define $f(0)=0$.

37. $\int_1^\infty\frac{dx}{x^{3/2}}=\lim_{b\to\infty}\int_1^b\frac{dx}{x^{3/2}}=\lim_{b\to\infty}\left(\frac{-2}{\sqrt{x}}\right)\Big|_1^b=2$

38. $\int_1^\infty\frac{dx}{x^2+7x+12}$

$$=\lim_{b\to\infty}\int_1^b\frac{dx}{x^2+7x+12}$$

$$=\lim_{b\to\infty}\left(-\ln\left(\frac{|x+4|}{|x+3|}\right)\right)\Bigg|_1^b=\ln\frac{5}{4}$$

39. $\int_{-\infty}^{-1}\frac{3dx}{3x-x^2}$

$$=\lim_{b\to-\infty}\int_b^{-1}\frac{3dx}{3x-x^2}$$

$$=\lim_{b\to-\infty}\left(-\ln\left(\frac{|x-3|}{|x|}\right)\right)\Bigg|_b^{-1}$$

$$=-2\ln(2)$$

40. $\int_0^3\frac{dx}{\sqrt{9-x^2}}=\lim_{b\to 3}\int_0^3\frac{dx}{\sqrt{9-x^2}}=\lim_{b\to 3}\left(\sin^{-1}\frac{x}{3}\right)\Big|_0^b=\frac{\pi}{2}$

41. $\int_0^1\ln(x)\,dx=\lim_{b\to 0}\int_b^1\ln(x)\,dx=\lim_{b\to 0}(x\ln x-x)\Big|_b^1=-1$

42. $\int_{-1}^1\frac{dy}{y^{2/3}}=\lim_{b\to 0^-}\int_{-1}^b\frac{dy}{y^{2/3}}+\lim_{b\to 0^+}\int_b^1\frac{dy}{y^{2/3}}$

$$=\lim_{b\to 0^1}\left(\frac{-2}{\sqrt{y}}\right)\Bigg|_{-1}^b+\lim_{b\to 0^+}\left(\frac{-2}{\sqrt{y}}\right)\Bigg|_b^1=6$$

43. $\int_{-2}^0\frac{d\theta}{(\theta+1)^{3/5}}$

$$=\lim_{b\to -1^-}\int_{-2}^b\frac{d\theta}{(\theta+1)^{3/5}}+\lim_{b\to -1^+}\int_b^0\frac{d\theta}{(\theta+1)^{3/5}}$$

$$=\lim_{b\to -1^{-1}}\left(\frac{5}{2}(x+1)^{\frac{2}{5}}\right)\Bigg|_{-2}^b+\lim_{b\to -1^+}\left(\frac{5}{2}(x+1)^{\frac{2}{5}}\right)\Bigg|_b^0=0$$

44. $\int_3^\infty\frac{2dx}{x^2-2x}=\lim_{b\to\infty}\int_3^b\frac{2dx}{x^2-2x}$

$$=\lim_{b\to\infty}\left(\text{In}\left(\frac{|x-2|}{|x|}\right)\right)\Bigg|_3^b=\text{In }3$$

45. $\int_0^\infty x^2e^{-x}dx=\lim_{b\to\infty}\int_0^b x^2e^{-x}dx$

$$=\lim_{b\to\infty}((-x^2-2x-2)e^{-x})\Big|_0^b=2$$

46. $\int_{-\infty}^0 xe^{3x}dx$

$$=\lim_{b\to-\infty}\int_b^0 xe^{3x}dx=\lim_{b\to-\infty}\left(\left(\frac{x}{3}-\frac{1}{9}\right)e^{3x}\right)\Bigg|_b^0=-\frac{1}{9}$$

47. $\int_{-\infty}^\infty\frac{dx}{e^x+e^{-x}}=\lim_{b\to-\infty}\int_b^0\frac{dx}{e^x+e^{-x}}+\lim_{b\to\infty}\int_0^b\frac{dx}{e^x+e^{-x}}$

$$=\lim_{b\to-\infty}(\tan^{-1}e^x)\Big|_0^b$$

47. Continued

$$+\lim_{b\to-\infty}(\tan^{-1}(e^x))\Big|_0^b$$

$$=\frac{\pi}{2}$$

48. $\int_{-\infty}^{\infty}\frac{4dx}{x^2+16}=\lim_{b\to-\infty}\int_b^0\frac{4dx}{x^2+16}+\lim_{b\to\infty}\int_0^b\frac{4dx}{x^2+16}$

$$=\lim_{b\to-\infty}\left(\tan^{-1}\left(\frac{x}{4}\right)\right)\Big|_b^0$$

$$+\lim_{b\to-\infty}\left(\tan^{-1}\left(\frac{x}{4}\right)\right)\Big|_0^b$$

$$=\pi$$

49. $0\le\int_1^{\infty}\frac{\ln z}{z}dz\le\int_1^{\infty}\ln z\,dz$

$$\int_1^{\infty}\ln z\,dz=\lim_{b\to\infty}\int_1^b\ln z\,dz=\lim_{b\to\infty}(z\ln z-z)\Big|_1^b=\infty$$

$\int_1^{\infty}\frac{\ln z}{z}dz$ diverges

50. $0\le\int_1^{\infty}\frac{e^{-t}}{\sqrt{t}}dt\le\int_1^{\infty}e^{-t}\,dt$

$$\int_1^{\infty}e^{-t}\,dt=\lim_{b\to\infty}\int_1^b e^{-t}\,dt=\lim_{b\to\infty}(-e^{-t})\Big|_1^b=-0+e=e^{-1}$$

$\int_1^{\infty}\frac{e^{-t}}{\sqrt{t}}dt$ converges

51. (a) $\left(\frac{-3/8}{3}\right)^{1/3}=1/2$

$$-3\left(\frac{1}{1/2}\right)=-6$$

(b) $\frac{1}{2}$

(c) $a_n=-6\left(\frac{1}{2}\right)^n$

$$a_n=-3(2^{2-n})$$

52. (a) $\left(\frac{5.5-11.5}{4}\right)=-1.5$

$$11.5-(-1.5)=13$$

(b) -1.5

(c) $a_n=13+(n-1)(-1.5)$

$$a_n=-1.5+14.5$$

53. (a) $\int_{-\infty}^{\infty}e^{-2|x|}\,dx=\lim_{b\to-\infty}\int_b^0 e^{2x}\,dx+\lim_{b\to\infty}\int_0^b e^{-2x}\,dx$

(b) $\lim_{b\to-\infty}\int_b^0 e^{2x}\,dx+\lim_{b\to\infty}\int_0^b e^{-2x}\,dx$

$$=\lim_{b\to-\infty}\left(\frac{e^{-2x}}{2}\right)\Big|_b^0+\lim_{b\to\infty}\left(\frac{e^{-2x}}{2}\right)\Big|_0^b=0+\frac{1}{2}+0+\frac{1}{2}=1$$

54. For $x\ge0, y\ge0$ on (0,1].

$$\text{Volume}=\int_0^1\pi(-\ln x)^2\,dx$$

$$=\pi\int_0^1(\ln x)^2\,dx$$

$$=\pi\lim_{b\to0^+}\int_b^1(\ln x)^2\,dx$$

Evaluate $\int(\ln x)^2\,dx$ by integration by parts.

$u=(\ln x)^2 \qquad dv=dx$

$du=\frac{2(\ln x)}{x}dx \quad v=x$

$$\int(\ln x)^2\,dx=x(\ln x)^2-\int 2\ln x\,dx$$

Evaluate $\int 2\ln x\,dx$ using integration by parts.

$dv=dx$

$u=2\ln x$

$du=\frac{2}{x}dx \quad v=x$

$$\int 2\ln x\,dx=2x\ln x-\int 2\,dx=2x\ln x-2x+C$$

$$\int(\ln x)^2\,dx=x(\ln x)^2-2x\ln x+2x+C$$

$$\text{Area}=\pi\lim_{b\to0^+}\left[x(\ln x)^2-2x\ln x+2x\right]_b^1$$

$$=\pi\lim_{b\to0^+}\left[2-b(\ln b)^2+2b\ln b-2b\right]$$

$$=2\pi-\lim_{b\to0^+}\frac{\pi(\ln b)^2}{1/b}+2\lim_{b\to0^+}\frac{\pi\ln b}{1/b}$$

$$=2\pi-\lim_{b\to0^+}\frac{2\pi(\ln b)(1/b)}{-1/b^2}+2\lim_{b\to0^+}\frac{\pi/b}{-1/b^2}$$

$$=2\pi-\lim_{b\to0^+}\frac{2\pi(\ln b)}{-1/b}+2\lim_{b\to0^+}(-\pi b)$$

$$=2\pi-\lim_{b\to0^+}\frac{2\pi/b}{1/b^2}-\lim_{b\to0^+}2\pi b=2\pi$$

55. For $x\ \ge 0, y\ \ge\ 0$ on $[0,\ \infty)$.

$$\text{Area}=\int_0^{\infty}xe^{-x}\,dx=\lim_{b\to\infty}\int_0^b xe^{-x}\,dx$$

Evaluate $\int xe^{-x}\,dx$ by using integration by parts.

$u=x \qquad dv=e^{-x}\,dx$

$du=dx \quad v=-e^{-x}$